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On Computing the Worst-case H_∞ Performance of Lur'e Systems with Uncertain Time-invariant Delays

Thapana Nampradit and David Banjerdpongchai*

Department of Electrical Engineering, Faculty of Engineering, Chulalongkorn University, 254 Phayathai Road, Pathumwan, Bangkok 10330, Thailand
*E-mail: bdavid@chula.ac.th

Abstract. This paper presents a worst-case H_∞ performance analysis for Lur'e systems with time-invariant delays. The sufficient condition to guarantee an upper bound of the worst-case performance is developed based on a delay-partitioning Lyapunov-Krasovskii functional containing an integral of sector-bounded nonlinearities. Using Jensen inequality and S-procedure, the delay-dependent criterion is given in terms of linear matrix inequalities. In addition, we extend the method to compute an upper bound of the worst-case performance of Lur'e systems subject to norm-bounded uncertainties by using a matrix eliminating lemma. Numerical results show that our criterion provide the least upper bound on the worst-case H_∞ performance comparing to the criteria derived based on existing techniques.

Keywords: Worst-case H_∞ performance, Lur'e systems, time-invariant delay, Lyapunov-Krasovskii functional, linear matrix inequality (LMI).

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1. Introduction

Lur'e systems [1] are nonlinear systems described by linear dynamic systems with feedback through sector-bounded nonlinearities. Such nonlinearities can be used to capture many common characteristics such as saturation, dead zone and spring stiffness. In addition to nonlinearities, time delays are frequently encountered in dynamic systems. The detrimental effects of time delays on system stability and performance are well known. Therefore, the studies on Lur'e systems with time delays (LSTD) are of theoretical and practical importance.

In the recent years, many studies have applied Lyapunov-Krasovskii Theorem [2] to develop absolute stability analysis for LSTD. In particular, various types of Lyapunov-Krasovskii functional (LKF) are used to formulate sufficient conditions. We can classify the stability conditions into two categories. The first is delay-independent criteria [3, 4], which provide the sufficient condition regardless of time delays. The second is delay-dependent criteria [5–12], which use the information on the delay length to prove the stability. Delay-independent criteria are quite limited in providing conclusion for the systems whose stability depends on the time delay. Therefore, subsequent studies concentrate on developing delay-dependent criteria. In [5], a model transformation and a bounding technique [13] are used to formulate the delay-dependent stability criterion. The drawback is that the upper bound of the cross product terms may not be tight and can lead to a conservative criterion. Later, a free weighting matrices (FWM) approach proposed by [6] applies relationship between each term in the Leibniz-Newton formula to the stability criterion. Although a bounding technique is not required for this approach, it introduces some slack variables apart from the matrix variables in the LKF. Motivated by [14], Jensen inequality is employed to derive absolute stability criteria for LSTD [7, 8]. It is shown in [15] that the FWM approach and Jensen inequality produce the identical results and conservatism, but the latter technique requires less number of decision variables.

Recently, several researchers proposed novel methods to analyze the absolute stability of LSTD based on the discretization scheme [14, 16]. They include delay-decomposition approach [17], N-segmentation method [9, 12], and delay-dividing approach [10]. The principle of these methods is to divide an interval $[-h, 0]$ into N equidistant partitions, and the LKF is separated corresponding to each subinterval of delay. Then, the Jensen inequality or the FWM approach is utilized to formulate the stability criterion in terms of linear matrix inequalities (LMIs). These methods successfully reduce conservatism of the stability criteria comparing to previous techniques. Moreover, it is proved that the conservatism of the criteria can be further reduced by increasing the number of partitions [18]. Among these criteria, [9], [11] and [12] proposed the absolute stability analysis for LSTD with a time-invariant delay. In [9], the delay interval is partitioned into N equidistant fragments, and the stability criterion is formulated by using Jensen inequality. However, the LKF used in [9] does not contain an integral of nonlinearities, which is essential for stability analysis of LSTD. In [11], the delay interval is divided into two specific subintervals, namely, $[-h, -h/3]$ and $[-h/3, 0]$, and the criterion is developed by using integral-equality technique, which is, in fact, another form of FWM approach. Although the LKF based on [11] utilize the integral of nonlinearities, their delay interval is only divided into two fixed subintervals, which is too specific. Recently, [12] fulfilled this gap by developing an improved absolute stability by combining the delay partitioning approach with utilizing integral terms involving sector-bounded nonlinearities in the LKF. The numerical results confirm that the criterion in [12] provides substantial improvement comparing to those in [9] and [11] especially when the sector bound is comparatively large.

The worst-case H_∞ performance is defined by L_2-gain of the nonlinear systems. A Lyapunov-Krasovskii functional can be incorporated with an upper bound of the L_2-gain to calculate an upper bound of such worst-case H_∞ performance; see [19] for sector-bounded nonlinearities, and [20] for another type of uncertainties. To the best of our knowledge, there is a few works on how to compute an upper bound of the worst-case H_∞ performance for LSTD [21]. It is worth developing the worst-case H_∞ performance criterion based on the combination of delay partitioning approach and employing the Lyapunov functional terms involving integral of nonlinearities. With the new performance analysis criterion, we can approach the actual value of the worst-case H_∞ performance.

The objective of this paper is to develop an effective method to compute an upper bound of the worst-case H_∞ performance of LSTD. The contribution of the paper is twofold. First, we give a method to analyze the worst-case H_∞ performance of nominal LSTD. The LKF incorporates with the integral of nonlinearities and the performance analysis employs the delay partitioning technique. Afterwards, Jensen inequality is applied to determine the performance criterion. Second, we extend the method to compute an upper bound of the worst-case H_∞ performance of LSTD with norm-bounded uncertainties. The criterion is formulated in terms of LMI by eliminating an uncertain matrix. In addition, we also develop two performance analysis criteria along with the concept presented in [9] and [11], and use them as the comparative criteria.

The paper is organized as follows. Section 2 introduces the notations and reviews the relevant lemmas. The definition of the worst-case H_∞ performance and analysis are first stated in section 3. In section 4, the worst-case performance criteria for both nominal and uncertain LSTD are presented. Section 5 shows the numerical results and compares the upper bounds of the worst-case H_∞ performance between the proposed criterion and the comparative criteria. Finally, Section 6 provides the summary of the main results and gives conclusions.

2. Preliminaries

$\mathbf{R}_+$ is the set of nonnegative numbers, and $\mathbf{R}^m$ is the set of real m-vectors. $\mathbf{1}$ and $\mathbf{0}$ denote a vector with all entries one and a vector with all entries zero of appropriate order, respectively. $\mathbf{R}^{m\times n}$ is the vector space of $m \times n$ real matrices. For any matrix $A \in \mathbf{R}^{m\times n}$, A^T denotes its transpose. I and 0 are an identity matrix and a null matrix of appropriate dimensions, respectively. The notation $\mathrm{diag}(\cdot)$ is used for diagonal matrices. For symmetric matrices A and B, the notation $A > B$ $(A \geq B)$ means that matrix $A - B$ is positive definite (positive semi-definite). Furthermore, for an arbitrary matrix C, and two symmetric matrices A and B, the symmetric term in a symmetric block matrix is denoted by *, *i.e.*,

$$\begin{bmatrix} A & C \\ * & B \end{bmatrix} := \begin{bmatrix} A & C \\ C^T & B \end{bmatrix}.$$

L_2^n is the Hilbert space of square-integrable signals defined over $\mathbf{R}_+$ with n-components; L_2^n is often abbreviated as L_2. The symbol $\|\cdot\|$ stands for the L_2 norm.

The notation ϕ represents a vector of nonlinearities, which belong to the set Φ characterized by memoryless, time-invariant nonlinearities satisfying certain sector conditions. In particular, given an input vector $\sigma := [\sigma_1, \ldots, \sigma_{n_p}]^T$, a lower bound vector $l := [l_1, \ldots, l_{n_p}]^T$ and an upper bound vector $m := [m_1, \ldots, m_{n_p}]^T$, with $l_i < m_i$ for all $i = 1, \ldots, n_p$, the set Φ can be described as follows.

$$\begin{aligned} \Phi(l, m) \quad := \quad & \{\phi : \mathbf{R}^{n_p} \to \mathbf{R}^{n_p} : \phi(\sigma) = \left[\phi_1(\sigma_1), \ldots, \phi_{n_p}(\sigma_{n_p})\right]^T, \\ & l_i\sigma_i^2 \leq \sigma_i\phi_i(\sigma_i) \leq m_i\sigma_i^2, \text{ for all } i = 1, \ldots, n_p\}. \end{aligned}$$

Finally, the following lemmas are useful for establishing the worst-case H_∞ performance criteria.

Lemma 1 (Jensen Inequality) [14] For any constant matrix $M \in \mathbf{R}^{m\times m}$, $M = M^T > 0$, scalar $\gamma > 0$, vector function $\omega : [0, \gamma] \to \mathbf{R}^m$ such that the integrations concerned are well defined, then

$$\gamma \int_0^\gamma \omega^T(\beta) M \omega(\beta) d\beta \geq \left(\int_0^\gamma \omega(\beta) d\beta\right)^T M \left(\int_0^\gamma \omega(\beta) d\beta\right).$$

Lemma 2 [22] Given matrices Q, H, E and R of appropriate dimensions and with Q and R symmetrical and $R > 0$, then

$$Q + HFE + E^T F^T H^T < 0,$$

for all F satisfying

$$F^T F \leq R,$$

if and only if there exists some $\epsilon > 0$ such that

$$Q + \epsilon H H^T + \epsilon^{-1} E^T R E < 0.$$

3. Problem Statement

We consider Lur'e systems with unknown time-invariant state delay described as follows.

$$\begin{array}{rcl} x(t) & = & Ax(t) + A_1 x(t-h) + B_p p(t) + B_w w(t), \\ q(t) & = & C_q x(t), \\ z(t) & = & C_z x(t), \\ p(t) & = & \phi(q(t)), \end{array} \tag{1}$$

with $\phi \in \Phi(\mathbf{0}, \mathbf{1})$ and zero initial condition $x(t) = 0, \forall t \in [-h, 0]$, $h \in \mathbf{R}_+$ is a time-invariant time delay in the state. The notation $x(t) \in \mathbf{R}^n$ is the state variable, $w(t) \in \mathbf{R}^{n_w}$ is the disturbance input which belongs to L_2, $z(t) \in \mathbf{R}^{n_z}$ is the performance output, $q(t) \in \mathbf{R}^{n_p}$, and $p(t) \in \mathbf{R}^{n_p}$ are the input/output of a vector mapping of sector-bounded nonlinearities denoted by ϕ. In addition, the pairs (A, B_p) and (C_q, A) are assumed to be *controllable* and *observable*, respectively. Next, the definitions of L_2-stability and the worst-case H_∞ performance for the system (1) are introduced.

Definition 1 (L_2-stable) A causal operator $H: \mathbf{R}^n \to \mathbf{R}^n$ is said to be L_2-*stable* if there exist $\gamma \geq 0$ and β such that

$$\| Hw \| \leq \gamma \| w \| + \beta, \quad \forall w \in L_2.$$

Definition 2 (Worst-case H_∞ Performance) Assume that the system (1) is L_2-*stable* with finite gain and zero bias. The worst-case H_∞ performance of the system (1) is defined by its L_2-*gain* described as follows.

$$J_\infty := \sup_{w \in L_2, w \neq 0} \frac{\|z\|}{\|w\|}, \tag{2}$$

where the supremum is taken over all nonzero output trajectories of the system (1) under zero initial condition.

While the actual value of J_∞ is difficult to compute, its *upper bound*, $\gamma_\infty \in \mathbf{R}_+$ such that $J_\infty \leq \gamma_\infty$, can be calculated from the following minimization problem [19].

$$\begin{array}{ll} \text{minimize} & \gamma_\infty^2, \\ \text{subject to} & V(x_t) + z^T(t)z(t) - \gamma_\infty^2 w^T(t)w(t) \leq 0, \end{array}$$

where $V(x_t)$ denotes the Lyapunov functional candidate. Therefore, the worst-case H_∞ performance analysis problem is to determine an upper bound of J_∞ of the system (1) for any time-invariant time delay $h \in (0, \bar{h}]$, *i.e.*, for a given $\bar{h}$, determine $\gamma_\infty \in \mathbf{R}_+$ such that $J_\infty \leq \gamma_\infty$.

In addition, we consider the LSTD with norm-bounded uncertainties described as follows.

$$\begin{array}{rcl} x(t) & = & [A + \Delta A(t)]x(t) + [A_1 + \Delta A_1(t)]x(t-h) + [B_p + \Delta B_p(t)]p(t) + B_w w(t), \\ q(t) & = & C_q x(t), \\ z(t) & = & C_z x(t), \\ p(t) & = & \phi(q(t)), \end{array} \tag{3}$$

with $\phi \in \Phi(\mathbf{0}, \mathbf{1})$. The uncertain variables $\Delta A(t)$, $\Delta A_1(t)$, and $\Delta B_p(t)$ are time-varying, but norm-bounded. The uncertainties are assumed to be of the following form

$$\begin{array}{rcl} \Delta A(t) & = & DF(t)E_0, \\ \Delta A_1(t) & = & DF(t)E_1, \\ \Delta B_p(t) & = & DF(t)E_2, \end{array} \tag{4}$$

where D, E_0, E_1, and E_2 are known constant real matrices of appropriate dimensions, and represent the structure of uncertainties, and $F(t)$ is an unknown matrix function with *Lebesgue measurable elements* satisfying the constraint $F^T(t)F(t) \leq I$. Similar to the nominal system (1), the worst-case H_∞ performance analysis problem for the uncertain system (3) is to determine an upper bound of J_∞ of the system (3) with any time-invariant time delay $h \in (0, \overline{h}]$, and for given matrices corresponding to norm-bounded uncertainties, *i.e.*, D, E_0, E_1, and E_2.

4. Worst-case H_∞ Performance Analysis

In this section, we present a sufficient condition for computing an upper bound of the worst-case H_∞ performance, which is derived by means of a Lyapunov-Krasovskii functional. Accordingly, the choice of LKF candidate plays a crucial role in developing the computing criterion. We employ the delay partitioning technique to the LKF for the system (1). The idea of this method is to divide the interval $[-h, 0]$ into N number of partitions, *i.e.*, $[-h, -h+r], [-h+r, -h+2r], \dots, [-r, 0]$, where $r = h/N$, and separately define the LKF involving delay on each subinterval. Consider the LKF candidate of the form

$$V(x_t) := V_1 + V_2 + V_3 + V_4, \tag{5}$$

with

$$\begin{aligned}
V_1 &= x^T(t)Px(t), \\
V_2 &= 2\sum_{i=1}^{n_p} \lambda_i \int_0^{q_i} \phi_i(\sigma)d\sigma, \\
V_3 &= \sum_{k=1}^{N} \int_{t-kr}^{t-(k-1)r} x^T(\xi)Q_k x(\xi)d\xi, \\
V_4 &= r\sum_{k=1}^{N} \int_{-kr}^{-(k-1)r} \int_{t+\theta}^{t} x^T(\xi)R_k x(\xi)d\xi d\theta,
\end{aligned}$$

where $P, Q_1, \dots, Q_N$, and $R_1, \dots, R_N$ are positive definite symmetric matrices of dimension $n \times n$, scalars $\lambda_1, \dots, \lambda_{n_p}$ are non-negative, and x_t denotes a piece of trajectory $x(t+\theta)$ for $-h \leq \theta \leq 0$. Next, we will show how to calculate the upper bound γ_∞ for the nominal LSTD (1).

Theorem 1 For a given $\overline{h} \in \mathbf{R}_+$, an upper bound of the worst-case H_∞ performance of LSTD (1) for any time-invariant time delay $h \in (0, \overline{h}]$ can be computed by minimizing γ_∞^2 subject to the constraint (6) over symmetric matrices $P > 0$, $Q_k > 0$, $R_k > 0$ for all $k = 1, \dots, N$, and diagonal matrices $\Lambda \geq 0$, $T \geq 0$.

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ * & \Psi_{22} & 0 & \Psi_{24} \\ * & * & \Psi_{33} & 0 \\ * & * & * & \Psi_{44} \end{bmatrix} \leq 0, \tag{6}$$

where

$$\Psi_{11} = \begin{bmatrix} \begin{pmatrix} PA + A^T P \\ +Q_1 - R_1 \\ +C_z^T C_z \end{pmatrix} & PA_1 & \begin{pmatrix} PB_p + C_q^T T \\ +A^T C_q^T \Lambda \end{pmatrix} \\ * & \begin{pmatrix} -Q_N \\ -R_N \end{pmatrix} & A_1^T C_q^T \Lambda \\ * & * & \begin{pmatrix} \Lambda C_q B_p - 2T \\ +B_p^T C_q^T \Lambda \end{pmatrix} \end{bmatrix}, \qquad \Psi_{12} = \begin{bmatrix} PB_w \\ 0 \\ \Lambda C_q B_w \end{bmatrix},$$

$$\Psi_{13} = \begin{bmatrix} R_1 & 0_{n\times(N-2)n} \\ 0 & 0_{n\times(N-2)n} \\ 0 & 0_{n\times(N-2)n} \end{bmatrix} + \begin{bmatrix} 0_{n\times(N-2)n} & 0 \\ 0_{n\times(N-2)n} & R_N \\ 0_{n\times(N-2)n} & 0 \end{bmatrix}, \qquad \Psi_{14} = \begin{bmatrix} \frac{\bar{h}}{N} A^T \sum_{k=1}^{N} R_k \\ \frac{\bar{h}}{N} A_1^T \sum_{k=1}^{N} R_k \\ \frac{\bar{h}}{N} B_p^T \sum_{k=1}^{N} R_k \end{bmatrix},$$

$$\Psi_{22} = -\gamma_\infty^2 I, \qquad \Psi_{24} = \frac{\bar{h}}{N} B_w^T \sum_{k=1}^{N} R_k, \qquad \Psi_{33} = \begin{bmatrix} \Gamma_1 & R_2 & & \\ * & \Gamma_2 & \ddots & \\ & \ddots & \ddots & R_{N-1} \\ & & * & \Gamma_{N-1} \end{bmatrix}, \qquad \Psi_{44} = -\sum_{k=1}^{N} R_k,$$

with $\Gamma_k = -Q_k + Q_{k+1} - R_k - R_{k+1}$.

Proof: Assume that $A + A_1$ is *Hurwitz*, and system matrices (A, B_p, C_q) are *minimal realization*. If there exists an LKF of the form (5) and $\gamma_\infty \in \mathbf{R}_+$ such that

$$V(x_t) + z^T(t)z(t) - \gamma_\infty^2 w^T(t)w(t) \le 0, \tag{7}$$

for all x_t satisfying the system equations (1), then $J_\infty \le \gamma_\infty$. Therefore, we seek P, $Q_1, \dots, Q_N$, $R_1, \dots, R_N$, and $\Lambda := \mathrm{diag}(\lambda_1, \dots, \lambda_{n_p})$ such that the constraint (7) is satisfied for all nonzero x_t satisfying (1) with a set of sector-bounded conditions

$$0 \le q_i(t)\phi_i(q_i(t)) \le q_i^2(t), \quad \forall i = 1, \dots, n_p. \tag{8}$$

To verify (7) under the set of constraints (8), we apply S-procedure [23] to establish the *sufficient condition* as follows.

$$V(x_t) + z^T(t)z(t) - \gamma_\infty^2 w^T(t)w(t) - \sum_{i=1}^{n_p} \tau_i\, q_i(t)(p_i(t) - q_i(t)) \le 0, \tag{9}$$

where $\tau_1 \ge 0, \dots, \tau_{n_p} \ge 0$. Note that for the case of single nonlinearity ($n_p = 1$), S-procedure is lossless, and the condition (9) is not only sufficient but also *necessary* for (7). By defining $T := \mathrm{diag}(\tau_1, \dots, \tau_{n_p})$, the inequality (9) can be written in vector-matrix notation as follows.

$$V(x_t) + z^T(t)z(t) - \gamma_\infty^2 w^T(t)w(t) - 2p^T(t)Tp(t) + 2x^T(t)C_q^T Tp(t) \le 0. \tag{10}$$

The derivative of each term in the LKF (5) with respect to time along the solution of (1) is given by

$$\begin{aligned}
V_1 &= x^T(t)\left[A^T P + PA\right]x(t) + 2x^T(t)PA_1x(t-h) + 2x^T(t)PB_pp(t) + 2x^T(t)PB_ww(t), \\
V_2 &= 2x^T(t)A^TC_q^T\Lambda p(t) + 2x^T(t-h)A_1^TC_q^T\Lambda p(t) + p^T(t)\left[B_p^TC_q^T\Lambda + \Lambda C_qB_p\right]p(t) + p(t)\Lambda C_qB_ww(t), \\
V_3 &= x^T(t)Q_1x(t) - x^T(t-h)Q_Nx(t-h) + \sum_{k=1}^{N-1}\left[x^T(t-kr)(-Q_k + Q_{k+1})x(t-kr)\right], \\
V_4 &= x^T(t)\sum_{k=1}^{N}\left(r^2R_k\right)x(t) - r\sum_{k=1}^{N}\int_{t-kr}^{t-(k-1)r} x^T(\xi)R_kx(\xi)d\xi.
\end{aligned}$$

Note that V_4 cannot be formulated in terms of LMI. Then, we rewrite the first term of V_4 by using the state equation in (1) as the following.

$$x^T(t)\sum_{k=1}^{N}\left(r^2R_k\right)x(t) = \mathbf{x}_1^T\begin{bmatrix}A^T\\A_1^T\\B_p^T\\B_w^T\end{bmatrix}\sum_{k=1}^{N}\left(r^2R_k\right)\begin{bmatrix}A^T\\A_1^T\\B_p^T\\B_w^T\end{bmatrix}^T\mathbf{x}_1, \tag{11}$$

where $\mathbf{x}_1 := \begin{bmatrix}x^T(t) & x^T(t-h) & p^T(t) & w^T(t)\end{bmatrix}^T$, and employ Lemma 1 (Jensen inequality) to bound the integral terms appeared in V_4 as follows.

$$\begin{aligned}
-r\sum_{k=1}^{N}\int_{t-kr}^{t-(k-1)r}x^T(\xi)R_kx(\xi)d\xi \quad &\leq \quad \sum_{k=1}^{N}\left(\int_{t-kr}^{t-(k-1)r}x(\xi)d\xi\right)^T(-R_k)\left(\int_{t-kr}^{t-(k-1)r}x(\xi)d\xi\right),\\
&= \quad \sum_{k=1}^{N}[x(t-(k-1)r)-x(t-kr)]^T(-R_k)\times\\
&\qquad [x(t-(k-1)r)-x(t-kr)],\\
&= \quad 2\mathbf{x}_1^T\Psi_{13}\mathbf{x}_2 + \mathbf{x}_1^T\begin{bmatrix}-R_1 & 0 & 0\\ * & -R_N & 0\\ * & * & 0\end{bmatrix}\mathbf{x}_1\\
&+ \quad \mathbf{x}_2^T\begin{bmatrix}-R_1-R_2 & R_2 & & \\ * & -R_1-R_2 & \ddots & \\ & \ddots & \ddots & R_{N-1}\\ & & * & -R_{N-1}-R_N\end{bmatrix}\mathbf{x}_2,
\end{aligned} \tag{12}$$

where $\mathbf{x}_2 := \begin{bmatrix}x^T(t-r) & \cdots & x^T(t-h+r)\end{bmatrix}^T$, and the entries left blank are zero. Substituting V_1, V_2, V_3 and V_4 into inequality (10), and applying (11) and the upper bound (12) for V_4, the sufficient condition for (10) is given as follows.

$$\mathbf{x}^T\Psi(h)\mathbf{x} \leq 0, \tag{13}$$

where $\mathbf{x} := \begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}$, and

$$\Psi(h) := \begin{bmatrix}\Psi_{11} & \Psi_{12} & \Psi_{13}\\ * & \Psi_{22} & 0\\ * & * & \Psi_{33}\end{bmatrix} - \begin{bmatrix}\Psi_{14}\\ \Psi_{24}\\ 0\end{bmatrix}\Psi_{44}^{-1}\begin{bmatrix}\Psi_{14}\\ \Psi_{24}\\ 0\end{bmatrix}^T. \tag{14}$$

Inequality (13) holds for all $\mathbf{x} \neq \mathbf{0}$ if and only if the following matrix inequality is satisfied.

$$\Psi(h) \leq 0. \tag{15}$$

Lastly, applying Schur complement [19, pp.7–8] and substituting h with $\overline{h}$, we obtain inequality (6).

It is important to note that the sufficient condition (6) guarantees an upper bound γ_∞ for the case of $h = \overline{h}$. Next, we show that (6) also guarantees the same γ_∞ for the system (1) for any time-invariant time delay $h \in (0, \overline{h}]$. Let $h = \overline{h} - \Delta h$, where $0 < \Delta h < \overline{h}$. Clearly, h lies in the interval $(0, \overline{h})$. By substituting h with $\overline{h} - \Delta h$, and isolating all terms involving Δh, the matrix inequality (14) becomes

$$\Psi(h) = \Psi(\overline{h}) + \frac{(2\Delta h\overline{h} - \Delta h^2)}{N^2}\begin{bmatrix}\Psi_{14}\\ \Psi_{24}\\ 0\end{bmatrix}\Psi_{44}^{-1}\begin{bmatrix}\Psi_{14}\\ \Psi_{24}\\ 0\end{bmatrix}^T \leq 0. \tag{16}$$

We observe that $(2\Delta h\overline{h} - \Delta h^2)/N^2 > 0$ and $\Psi_{44} < 0$. Then, the isolated terms are always negative. Thus, if $\Psi(\overline{h}) \leq 0$ holds, $\Psi(h) \leq 0$ also holds. In other words, the matrix inequality $\Psi(\overline{h}) \leq 0$ implies (16), and

Theorem 1 guarantees an upper bound γ_∞ for the systems (1) for any time-invariant time delay $h \in (0, \bar{h}]$. This completes the proof. ■

It is appeared that the condition (6) is LMI over matrix variables P, $Q_1, \dots, Q_N$, $R_1, \dots, R_N$, Λ, and T for a given $\bar{h} \in \mathbf{R}_+$. Hence, the problem of minimizing γ_∞^2 subject to (6) can be cast as a minimization problem with LMI constraints which can be solved efficiently.

Remark 1 In Theorem 1, it is straightforward to handle general sector condition $\phi \in \Phi(l, m)$. By using *loop transformation* [24], LSTD (1) with $\phi \in \Phi(l, m)$ can be transformed to an equivalent LSTD with $\bar{\phi} \in \Phi(\mathbf{0}, \mathbf{1})$. In particular, define

$$\bar{\phi}_i\big(q_i(t)\big) := \frac{1}{m_i - l_i}\big[\phi_i(q_i(t)) - l_i q_i(t)\big].$$

It is easy to show that $0 \le \sigma_i \bar{\phi}_i(\sigma_i) \le \sigma_i^2$ for all $i = 1, \dots, n_p$, *i.e.*, $\bar{\phi} \in \Phi(\mathbf{0}, \mathbf{1})$. Let $L := \text{diag}(l)$, $M := \text{diag}(m)$, and $\bar{p}(t) := (M - L)^{-1}(p(t) - Lq(t))$. We then substitute $p(t) = (M - L)\bar{p}(t) + Lq(t)$ into (1), and obtain the equivalent LSTD systems as follows.

$$\begin{array}{rcl} x(t) &=& \bar{A}x(t) + A_1 x(t-h) + \bar{B}_p \bar{p}(t) + B_w w(t), \\ q(t) &=& C_q x(t), \\ z(t) &=& C_z x(t), \\ \bar{p}(t) &=& \bar{\phi}(q(t)), \end{array} \tag{17}$$

with $\bar{\phi} \in \Phi(\mathbf{0}, \mathbf{1})$, where $\bar{A} = A + B_p L C_q$ and $\bar{B}_p = B_p(M - L)$. Note that the transformed LSTD (17) is equivalent to the original LSTD (1), and we can calculate an upper bound γ_∞ for LSTD (1) by considering the system (17).

Next, we will derive another sufficient condition for computing an upper bound of J_∞ for the uncertain LSTD (3).

Theorem 2 For a given $\bar{h} \in \mathbf{R}_+$, an upper bound of the worst-case H_∞ performance of the systems (3) for any time-invariant time delay $h \in (0, \bar{h}]$ can be computed by minimizing γ_∞^2 subject to the constraint (18) over symmetric matrices $P > 0$, $Q_k > 0$, $R_k > 0$ for all $k = 1, \dots, N$, and diagonal matrices $\Lambda \ge 0$, $T \ge 0$, and a scalar variable $\epsilon > 0$.

$$\begin{bmatrix} \tilde{\Psi}_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \Psi_{15} \\ * & \Psi_{22} & 0 & \Psi_{24} & 0 \\ * & * & \Psi_{33} & 0 & 0 \\ * & * & * & \Psi_{44} & \Psi_{45} \\ * & * & * & * & \Psi_{55} \end{bmatrix} \le 0, \tag{18}$$

where

$$\tilde{\Psi}_{11} := \Psi_{11} + \epsilon \begin{bmatrix} E_0 \\ E_1 \\ E_2 \end{bmatrix} [E_0 \quad E_1 \quad E_2], \qquad \Psi_{15} = \begin{bmatrix} PD \\ 0 \\ \Lambda C_q D \end{bmatrix},$$

$$\Psi_{45} = \frac{\bar{h}}{N}\left(\sum_{k=1}^{N} R_k\right) D, \qquad \Psi_{55} = -\epsilon I,$$

Ψ_{11}, Ψ_{12}, Ψ_{13}, Ψ_{14}, Ψ_{22}, Ψ_{24}, Ψ_{33}, and Ψ_{44} are the same as defined in Theorem 1.

Proof: By applying Theorem 1 to the uncertain systems (3), the worst-case H_∞ performance criterion consists of the following LMI.

$$\Psi + \begin{bmatrix} \Psi_{15} \\ 0 \\ 0 \\ \Psi_{45} \end{bmatrix} F(t)[E_0 \quad E_1 \quad E_2 \quad 0 \quad 0] + [E_0 \quad E_1 \quad E_2 \quad 0 \quad 0]^T F^T(t) \begin{bmatrix} \Psi_{15} \\ 0 \\ 0 \\ \Psi_{45} \end{bmatrix}^T \leq 0, \tag{19}$$

where Ψ is defined as

$$\Psi := \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ * & \Psi_{22} & 0 & \Psi_{24} \\ * & * & \Psi_{33} & 0 \\ * & * & * & \Psi_{44} \end{bmatrix} \leq 0.$$

It follows from Lemma 2 that the matrix inequality (19) is true for all uncertain matrix $F(t)$ satisfying $F^T(t)F(t) \leq I$ if and only if there exists a scalar $\epsilon > 0$ such that

$$\Psi + \epsilon^{-1} \begin{bmatrix} \Psi_{15} \\ 0 \\ 0 \\ \Psi_{45} \end{bmatrix} \begin{bmatrix} \Psi_{15} \\ 0 \\ 0 \\ \Psi_{45} \end{bmatrix}^T + \epsilon \begin{bmatrix} E_0^T \\ E_1^T \\ E_2^T \\ 0 \\ 0 \end{bmatrix} [E_0 \quad E_1 \quad E_2 \quad 0 \quad 0] \leq 0. \tag{20}$$

Absorbing the last term into Ψ and applying the Schur complement [19], the matrix inequality (18) holds. This completes the proof. ■

Remark 2 It is observed that the terms involving ϵ and ϵ^{-1} in the matrix inequality (20) are all positive. Then, the feasible set for (18) is smaller than that of (6), and the γ_∞ obtained for the uncertain LSTD (3) should be greater than or at least equal to that for the nominal LSTD.

In order to illustrate the effectiveness of the proposed criteria, we develop worst-case H_∞ performance analysis criteria for uncertain LSTD along with the N-segmentation technique in [9] with $N = 3$ and the integral-equality approach in [11] as stated in Theorem 3 and Theorem 4, respectively.

Theorem 3 (Extension of Wu et al. (2009)) For a given $\bar{h} \in \mathbf{R}_+$, an upper bound of the worst-case H_∞ performance of the systems (3) for any time-invariant time delay $h \in (0, \bar{h}]$ can be computed by minimizing γ_∞^2 subject to the constraint (21) over symmetric matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $R_1 > 0$, $R_2 > 0$, $R_3 > 0$, and scalar variables $\tau \geq 0$ and $\epsilon > 0$.

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} \\ * & \Xi_{22} & 0 & \Xi_{24} & 0 \\ * & * & \Xi_{33} & 0 & 0 \\ * & * & * & \Xi_{44} & \Xi_{45} \\ * & * & * & * & \Xi_{55} \end{bmatrix} \leq 0, \tag{21}$$

where

$$\Xi_{11} = \begin{bmatrix} \begin{pmatrix} PA + A^T P \\ +Q_1 - R_1 \\ +C_z^T C_z + \epsilon E_0^T E_0 \end{pmatrix} & \begin{pmatrix} PA_1 \\ +\epsilon E_0^T E_1 \end{pmatrix} & \begin{pmatrix} PB_p \\ +\tau C_q^T \\ +\epsilon E_0^T E_2 \end{pmatrix} \\ * & \begin{pmatrix} -Q_3 - R_3 \\ +\epsilon E_1^T E_1 \end{pmatrix} & \epsilon E_1^T E_2 \\ * & * & \begin{pmatrix} -2\tau \\ +\epsilon E_2^T E_2 \end{pmatrix} \end{bmatrix}, \quad \Xi_{12} = \begin{bmatrix} PB_w \\ 0 \\ 0 \end{bmatrix}, \quad \Xi_{13} = \begin{bmatrix} R_1 & 0 \\ 0 & R_3 \\ 0 & 0 \end{bmatrix},$$

$$\Xi_{14} = \begin{bmatrix} \frac{\bar{h}}{3} A^T (R_1 + R_2 + R_3) \\ \frac{\bar{h}}{3} A_1^T (R_1 + R_2 + R_3) \\ \frac{\bar{h}}{3} B_p^T (R_1 + R_2 + R_3) \end{bmatrix}, \qquad \Xi_{15} = \begin{bmatrix} PD \\ 0 \\ 0 \end{bmatrix}, \qquad \Xi_{22} = -\gamma_\infty^2 I, \qquad \Xi_{24} = \frac{\bar{h}}{3} B_w^T (R_1 + R_2 + R_3),$$

$$\Xi_{33} = \begin{bmatrix} \begin{pmatrix} -Q_1 + Q_2 \\ -R_1 - R_2 \end{pmatrix} & R_2 \\ * & \begin{pmatrix} -Q_2 + Q_3 \\ -R_2 - R_3 \end{pmatrix} \end{bmatrix}, \qquad \Xi_{44} = -R_1 - R_2 - R_3, \qquad \Xi_{45} = \frac{\bar{h}}{3} (R_1 + R_2 + R_3) D,$$

and $\Xi_{55} = -\epsilon I$.

Theorem 4 (Extension of Qiu & Zhang (2011)) For a given $\bar{h} \in \mathbf{R}_+$, an upper bound of the worst-case H_∞ performance of the systems (3) for any time-invariant time delay $h \in (0, \bar{h}]$ can be computed by minimizing γ_∞^2 subject to the constraint (22) over symmetric matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $R_1 > 0$, $R_2 > 0$, full matrices M_{11}, M_{12}, M_{13}, M_{14}, M_{21}, M_{22}, M_{23}, M_{24}, a diagonal matrix $\Lambda \geq 0$, and scalar variables $\tau \geq 0$ and $\epsilon > 0$.

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & \Theta_{15} & \Theta_{16} & \Theta_{17} \\ * & \Theta_{22} & \Theta_{23} & 0 & \Theta_{25} & \Theta_{26} & 0 \\ * & * & \Theta_{33} & \Theta_{34} & \Theta_{35} & \Theta_{36} & \Theta_{37} \\ * & * & * & \Theta_{44} & \Theta_{45} & 0 & 0 \\ * & * & * & * & \Theta_{55} & 0 & \Theta_{57} \\ * & * & * & * & * & \Theta_{66} & 0 \\ * & * & * & * & * & * & \Theta_{77} \end{bmatrix} \leq 0, \tag{22}$$

where

$$\Theta_{11} = PA + A^T P + Q_1 + M_{11} + M_{11}^T + C_z^T C_z + \epsilon E_0^T E_0,$$

$$\Theta_{12} = \begin{bmatrix} \begin{pmatrix} PA_1 + M_{12}^T \\ -M_{21} + \epsilon E_0^T E_1 \end{pmatrix} & \begin{pmatrix} -M_{11} + M_{13}^T \\ +M_{21} \end{pmatrix} \end{bmatrix}, \qquad \Theta_{13} = PB_p + A^T C_q^T \Lambda + \tau C_q^T + M_{14}^T + \epsilon E_0^T E_2,$$

$$\Theta_{14} = PB_w, \qquad \Theta_{15} = \frac{\bar{h}}{3} A^T R_1 + \frac{2\bar{h}}{3} A^T R_2, \qquad \Theta_{16} = \begin{bmatrix} \sqrt{\frac{\bar{h}}{3}} M_{11} & \sqrt{\frac{2\bar{h}}{3}} M_{21} \end{bmatrix}, \qquad \Theta_{17} = PD,$$

$$\Theta_{22} = \begin{bmatrix} \begin{pmatrix} -Q_2 - M_{22} \\ -M_{22}^T + \epsilon E_1^T E_1 \end{pmatrix} & \begin{pmatrix} -M_{12} + M_{22} \\ -M_{23}^T \end{pmatrix} \\ * & \begin{pmatrix} -Q_1 + Q_2 \\ -M_{13} - M_{13}^T \\ +M_{23} + M_{23}^T \end{pmatrix} \end{bmatrix}, \qquad \Theta_{23} = \begin{bmatrix} A_1^T C_q^T \Lambda - M_{24}^T + \epsilon E_1^T E_2 \\ -M_{14}^T + M_{24}^T \end{bmatrix},$$

$$\Theta_{25} = \begin{bmatrix} \frac{\bar{h}}{3} A_1^T R_1 + \frac{2\bar{h}}{3} A_1^T R_2 \\ 0 \end{bmatrix}, \qquad \Theta_{26} = \begin{bmatrix} \sqrt{\frac{\bar{h}}{3}} M_{12} & \sqrt{\frac{2\bar{h}}{3}} M_{22} \\ \sqrt{\frac{\bar{h}}{3}} M_{13} & \sqrt{\frac{2\bar{h}}{3}} M_{23} \end{bmatrix},$$

$$\Theta_{33} = \Lambda C_q B_p + B_p^T C_q^T \Lambda - 2\tau + \epsilon E_2^T E_2, \qquad \Theta_{34} = \Lambda C_q B_w, \qquad \Theta_{35} = \frac{\overline{h}}{3} B_p^T R_1 + \frac{2\overline{h}}{3} B_p^T R_2,$$

$$\Theta_{36} = \begin{bmatrix} \sqrt{\frac{\overline{h}}{3}} M_{14} & \sqrt{\frac{2\overline{h}}{3}} M_{24} \end{bmatrix}, \qquad \Theta_{37} = \Lambda C_q D, \qquad \Theta_{44} = -\gamma_\infty^2 I, \qquad \Theta_{45} = \frac{\overline{h}}{3} B_w^T R_1 + \frac{2\overline{h}}{3} B_w^T R_2,$$

$$\Theta_{55} = -\frac{\overline{h}}{3} R_1 - \frac{2\overline{h}}{3} R_2, \qquad \Theta_{57} = \frac{\overline{h}}{3} R_1 D + \frac{2\overline{h}}{3} R_2 D, \qquad \Theta_{66} = -\mathrm{diag}(R_1, R_2), \qquad \Theta_{77} = -\epsilon I.$$

Remark 3 The comparative criteria can be considered as special cases of the proposed criterion in Theorem 2, *i.e.*,

- The criterion presented in Theorem 3 is a special case of Theorem 2 when $N = 3$, $\Lambda = 0$, and $T = \tau I$ for $\tau \in \mathbf{R}_+$.
- Since FWM approach and Jensen inequality approach produce the stability criteria at the same level of conservatism [15], the criterion presented in Theorem 4, which utilizes a form of FWM approach, namely integral-equality approach, is as conservative as a special case of Theorem 2 when $N = 3$, $Q_2 = Q_3$, $R_2 = R_3$, and $T = \tau I$ for $\tau \in \mathbf{R}_+$.
- The criterion proposed in [21] is a special case of Theorem 2 when $N = 1$.

Clearly, the conservatism of the proposed criterion is less than or at least equal to those of the comparative criteria. The additional free variables can be potentially the key to establish the less conservative criterion.

4. Numerical Results

The conservatism of the proposed criterion in Theorem 2 with $N = 3$, the extension of [9] in Theorem 3, and the extension of [11] in Theorem 4 are compared on three numerical examples. The LMI Lab [25] which employs the projective interior-point method [26], is used for solving the LMI minimization problems in the experiments.

Example 1: Consider the system of the form (3) with the following parameters.

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \qquad B_p = \begin{bmatrix} -0.2 \\ -0.3 \end{bmatrix}, \qquad B_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C_q = [0.6 \quad 0.8], \qquad \mathrm{C_z} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \| \Delta A(t) \| \le \alpha, \qquad \| \Delta A_1(t) \| \le \alpha, \qquad \Delta B_p(t) = 0,$$

$$\phi \in \Phi(0.35 - \delta l, 0.35 + \delta l),$$

where δl characterizes the sector bound of nonlinearity, and α represents magnitude of the uncertainty. The system data is taken from [7] with slight modifications. The uncertainty model $\Delta A(t)$, $\Delta A_1(t)$, and $\Delta B_p(t)$ can be described by (4) with $D = \alpha I$, $E_0 = E_1 = I$, $E_2 = 0$, where $F^T(t)F(t) \le I$, $F(t) \in \mathbf{R}^{2\times 2}$. Loop transformation is applied so that LSTD with $\phi \in \Phi(0.35 - \delta l, 0.35 + \delta l)$ is transformed into an equivalent LSTD with $\overline{\phi} \in \Phi(0,1)$.

Taking $\delta l = 2$, $\alpha = 0.10$, and $\overline{h} = 1$, the minimal γ_∞ calculated using Theorem 2 is $\gamma_\infty = 2.7408$ with

$$P = \begin{bmatrix} 8.7313 & -1.4537 \\ -1.4537 & 2.0114 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 14.7252 & -2.8535 \\ -2.8535 & 0.6092 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 16.7276 & -2.7634 \\ -2.7634 & 1.4269 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 19.7737 & -3.1326 \\ -3.1326 & 2.4093 \end{bmatrix}, \qquad R_1 = \begin{bmatrix} 7.6115 & -0.7510 \\ -0.7510 & 3.4954 \end{bmatrix},$$

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$$R_2 = \begin{bmatrix} 12.4059 & -1.9330 \\ -1.9330 & 3.7336 \end{bmatrix}, \qquad R_3 = \begin{bmatrix} 13.1966 & -2.0507 \\ -2.0507 & 3.7298 \end{bmatrix},$$

$\Lambda = 1.5209, T = 3.0348$, and $\epsilon = 0.1970$ satisfying LMI (18). With the same parameters, the minimal γ_∞ computed using Theorem 3 is $\gamma_\infty = 4.7814$ for the following matrices

$$P = \begin{bmatrix} 11.1884 & -0.7095 \\ -0.7095 & 3.2712 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 23.9080 & -3.3888 \\ -3.3888 & 0.6270 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 25.7629 & -2.8128 \\ -2.8128 & 1.9839 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 42.9424 & -5.9540 \\ -5.9540 & 4.1816 \end{bmatrix}, \qquad R_1 = \begin{bmatrix} 3.7002 & 1.7172 \\ 1.7172 & 3.6697 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 3.2480 & 1.8696 \\ 1.8696 & 3.6196 \end{bmatrix}, \qquad R_3 = \begin{bmatrix} 31.0947 & -7.4719 \\ -7.4719 & 6.6949 \end{bmatrix},$$

$\tau = 7.3880$, and $\epsilon = 0.3914$ satisfying LMI (21). Finally, the minimal γ_∞ obtained from Theorem 4 is $\gamma_\infty = 2.9566$ with the matrices

$$P = \begin{bmatrix} 9.7988 & -1.6449 \\ -1.6449 & 2.1395 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 17.5776 & -3.3518 \\ -3.3518 & 0.7009 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 20.9623 & -3.5168 \\ -3.5168 & 2.1990 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} 3.5426 & -0.5459 \\ -0.5459 & 1.2692 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 4.6697 & -0.6705 \\ -0.6705 & 1.4094 \end{bmatrix},$$

$$M_{11} = \begin{bmatrix} -10.6766 & 1.5538 \\ 1.6812 & -3.3729 \end{bmatrix}, \qquad M_{12} = \begin{bmatrix} -0.0058 & -0.1668 \\ 0.0194 & 0.2759 \end{bmatrix}, \qquad M_{13} = \begin{bmatrix} 10.6648 & -1.4574 \\ -1.6801 & 3.2881 \end{bmatrix},$$

$$M_{14} = [-0.0207 \quad 0.0573], \qquad M_{21} = \begin{bmatrix} -0.0848 & 0.0580 \\ 0.2792 & 0.0222 \end{bmatrix}, \qquad M_{22} = \begin{bmatrix} 6.9071 & -0.9653 \\ -0.8352 & 2.1251 \end{bmatrix},$$

$$M_{23} = \begin{bmatrix} -6.8753 & 0.9309 \\ 0.6799 & -2.1358 \end{bmatrix}, \qquad M_{24} = [0.0159 \quad -0.0015],$$

$\Lambda = 1.7023$, $\tau = 3.2314$, and $\epsilon = 0.2153$ satisfying LMI (22).

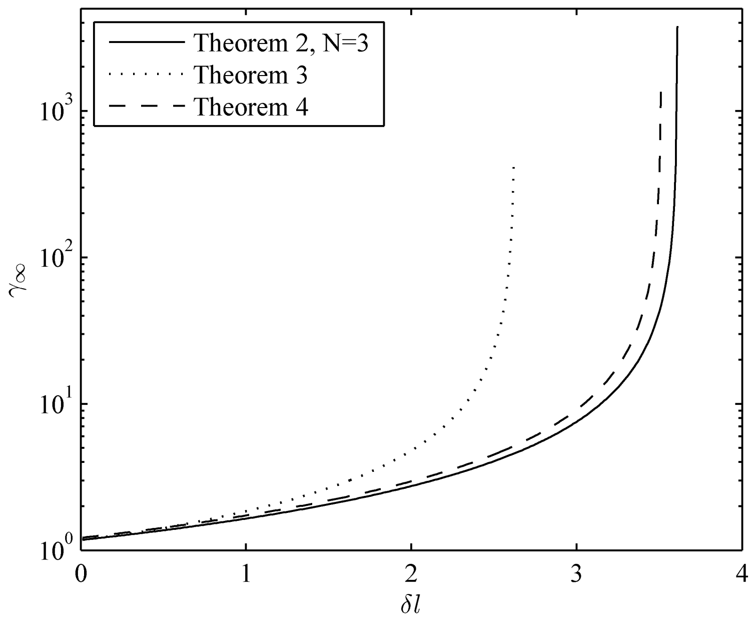

Fig. 1. Upper bounds of J_∞ for Ex. 1 when $\bar{h} = 1$ and $\alpha = 0.10$.

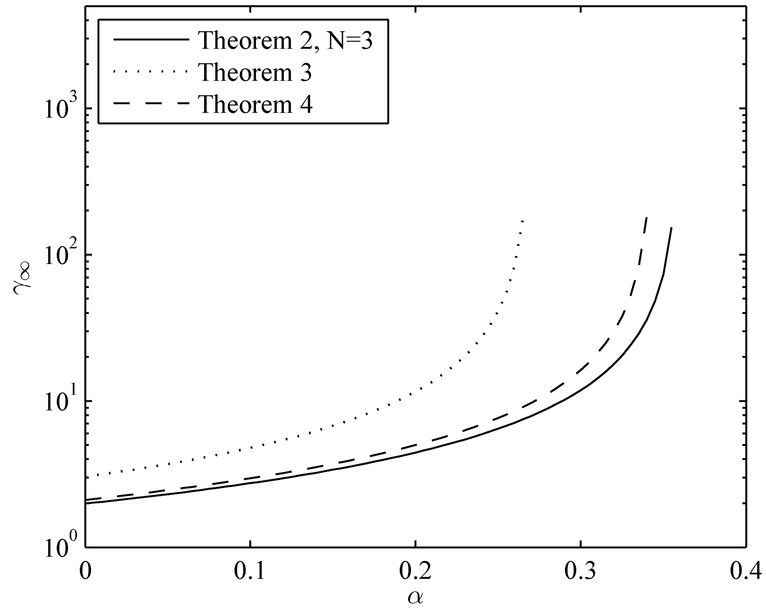

Fig. 2. Upper bounds of J_∞ for Ex. 1 when $\bar{h} = 1$ and $\delta l = 2$.

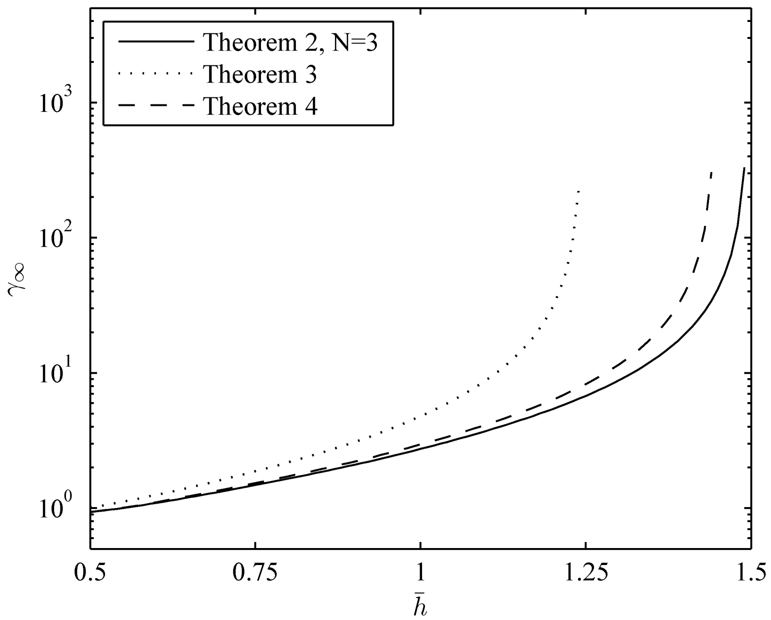

Fig. 3. Upper bounds of J_∞ for Ex. 1 when $\delta l = 2$ and $\alpha = 0.10$.

We compare γ_∞, calculated from three different methods for the system with various parameters δl, α, and $\bar{h}$. Figure 1 shows γ_∞, versus δl for Ex. 1 when $\bar{h}$ and α are fixed. It is observed that γ_∞, is increased as the sector bound δl is increased. Likewise, γ_∞, grows up as the uncertainty α, or the bound on time delay $\bar{h}$ is enlarged as shown in Fig. 2 and Fig. 3, respectively. Using the proposed criterion in Theorem 2 always gives the smaller γ_∞, when compared with those obtained from other comparative criteria, especially for large δl, α and $\bar{h}$. Moreover, the proposed criterion can guarantee γ_∞, for a wider range of parameters.

Example 2: Consider the uncertain LSTD of the form (3) with the following parameters.

$$A = \begin{bmatrix} -2.0 & -1.0 \\ 0.5 & 0.2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.5 & 1.0 \\ -0.1 & -0.8 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0.5 & 0 \\ 0 & -0.2 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C_q = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad C_z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \| \Delta A(t) \| \leq \alpha, \quad \| \Delta A_1(t) \| \leq \alpha, \quad \Delta B_p(t) = 0,$$

$$\phi \in \Phi\left(\begin{bmatrix} 0.25 - \delta l \\ 0.35 - \delta l \end{bmatrix}, \begin{bmatrix} 0.25 + \delta l \\ 0.35 + \delta l \end{bmatrix} \right).$$

This example is modified from the system given in [7]. Similar to the previous example, the norm-bounded uncertainty can be described by (4) with $D = \alpha I$, E_0 and E_1 are identity matrices of appropriate dimension, and E_2 is a null matrix, where $F^T(t)F(t) \leq I$, $F(t) \in \mathbf{R}^{2\times 2}$. Loop transformation is applied so that LSTD with ϕ defined above is transformed to an equivalent LSTD with $\overline{\phi} \in \Phi(\mathbf{0}, \mathbf{1})$.

For this example, we let $\delta l = 2$, $\alpha = 0.10$, and $\overline{h} = 1$. The minimal γ_∞ provided by using Theorem 2 is $\gamma_\infty = 6.3293$ with

$$P = \begin{bmatrix} 4.0176 & 1.3260 \\ 1.3260 & 5.2710 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 4.8497 & 0.7052 \\ 0.7052 & 0.1026 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 5.3397 & 0.6764 \\ 0.6764 & 0.0857 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 10.3549 & 1.3080 \\ 1.3080 & 0.1652 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 3.7892 & 6.2603 \\ 6.2603 & 31.1233 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.7885 & 4.9137 \\ 4.9137 & 30.6263 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 16.5208 & 11.3249 \\ 11.3249 & 32.8461 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 13.2721 & 0 \\ 0 & 15.2884 \end{bmatrix}, \quad T = \begin{bmatrix} 41.8920 & 0 \\ 0 & 8.2773 \end{bmatrix},$$

and $\epsilon = 0.3608$ satisfying LMI (18). Next, the minimal γ_∞ calculated using Theorem 3 is $\gamma_\infty = 9.1201$ with

$$P = \begin{bmatrix} 7.8496 & 1.9977 \\ 1.9977 & 7.7780 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 8.7277 & 0.6439 \\ 0.6439 & 0.0475 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 8.7275 & 0.6438 \\ 0.6438 & 0.0475 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 12.8981 & 0.9087 \\ 0.9087 & 0.0640 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 1.1332 & 6.0244 \\ 6.0244 & 32.1217 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 1.1303 & 6.0232 \\ 6.0232 & 32.1215 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 27.5940 & 13.5520 \\ 13.5520 & 33.7972 \end{bmatrix},$$

$\tau = 16.9354$, and $\epsilon = 0.5486$ satisfying LMI (21). Lastly, the minimal $\gamma_\infty = 7.7641$ is computed using Theorem 4 when LMI (22) is satisfied by

$$P = \begin{bmatrix} 3.5740 & 0.7211 \\ 0.7211 & 6.1568 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 7.8664 & -0.3733 \\ -0.3733 & 0.0177 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 7.8664 & -0.3733 \\ -0.3733 & 0.0177 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} 0.2079 & 1.6459 \\ 1.6459 & 13.0362 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.2078 & 1.6459 \\ 1.6459 & 13.0362 \end{bmatrix},$$

$$M_{11} = \begin{bmatrix} -0.6240 & -4.9404 \\ -4.9400 & -39.1275 \end{bmatrix}, \quad M_{12} = \begin{bmatrix} -0.0001 & -0.0009 \\ -0.0012 & -0.0095 \end{bmatrix}, \quad M_{13} = \begin{bmatrix} 0.6241 & 4.9412 \\ 4.9412 & 39.1370 \end{bmatrix},$$

$$M_{14} = \begin{bmatrix} 3.1227\times 10^{-5} & 2.2816\times 10^{-4} \\ -6.4550\times 10^{-5} & -5.1007\times 10^{-4} \end{bmatrix}, \quad M_{21} = \begin{bmatrix} 0.0001 & 0.0007 \\ 0.0006 & 0.0046 \end{bmatrix},$$

$$M_{22} = \begin{bmatrix} 0.3118 & 2.4693 \\ 2.4691 & 19.5567 \end{bmatrix}, \quad M_{23} = \begin{bmatrix} -0.3119 & -2.4697 \\ -2.4697 & -19.5613 \end{bmatrix},$$

$$M_{24} = \begin{bmatrix} -2.9267\times10^{-5} & -2.2698\times10^{-4} \\ -1.7514\times10^{-5} & -1.3881\times10^{-4} \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} 13.6751 & 0 \\ 0 & 21.1908 \end{bmatrix},$$

$\tau = 14.4800$, and $\epsilon = 0.4310$.

The computed γ_∞ versus δl, α, and $\overline{h}$ for Ex. 2 are shown in Figs. 4–6, respectively. It can be seen that Theorem 2 always give the less conservative result than those obtained from other criteria. In addition, Theorem 2 is capable of finding γ_∞ for a wider range of parameters, which indicate that the proposed criterion has the advantage over the comparative criteria.

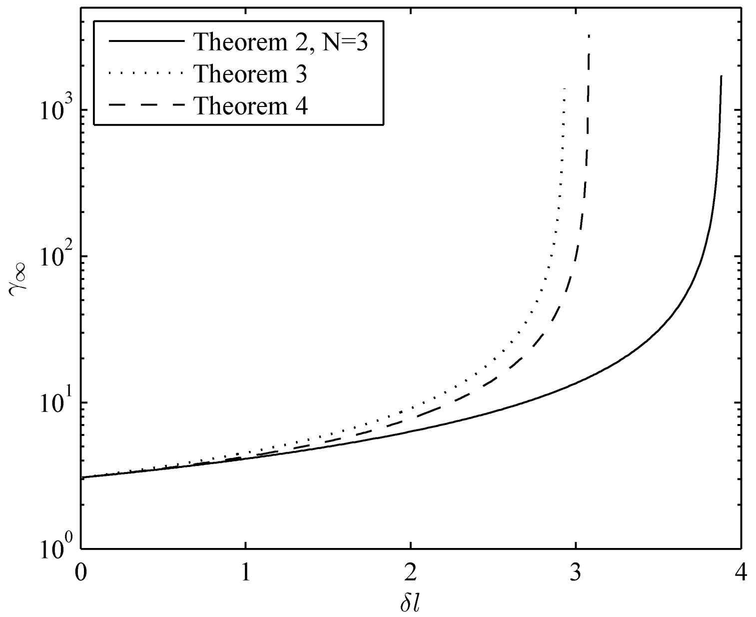

Fig. 4. Upper bounds of J_∞ for Ex. 2 when $\overline{h} = 1$ and $\alpha = 0.10$.

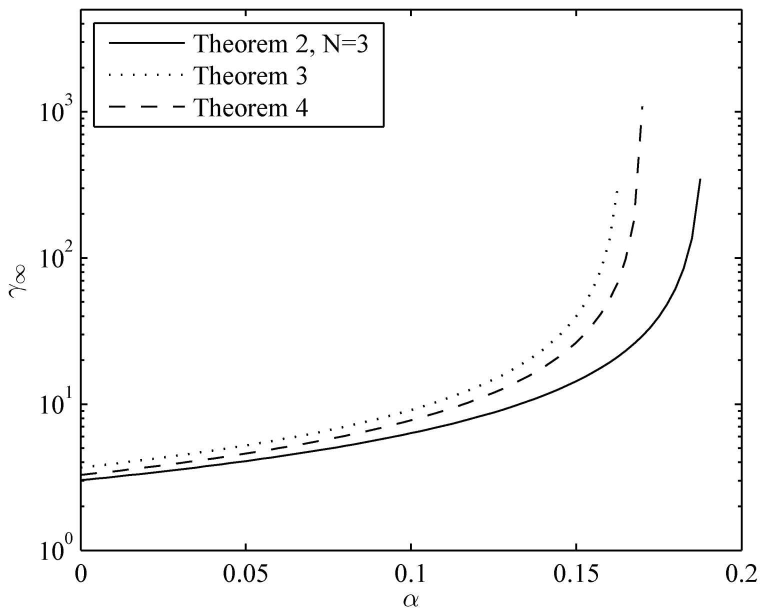

Fig. 5. Upper bounds of J_∞ for Ex. 2 when $\overline{h} = 1$ and $\delta l = 2$.

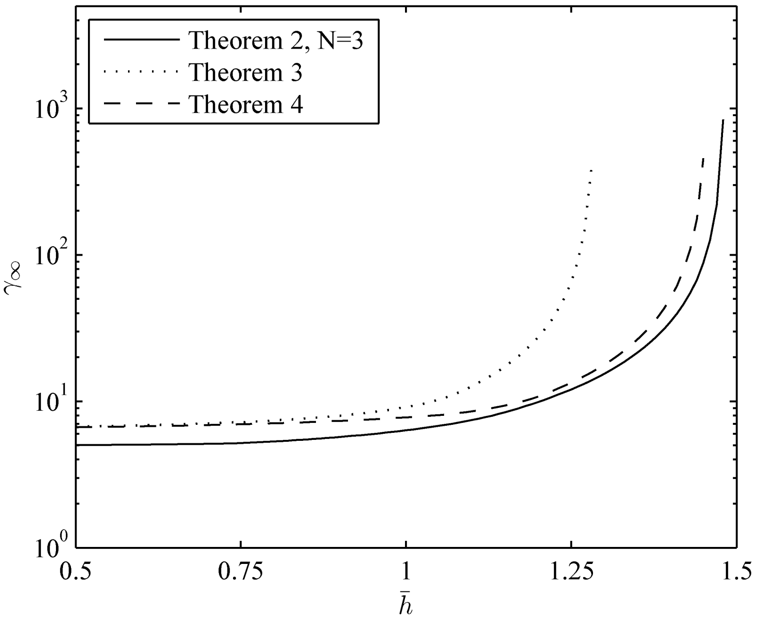

Fig. 6. Upper bounds of J_∞ for Ex. 2 when $\delta l = 2$ and $\alpha = 0.10$.

Example 3: Consider the uncertain system of the form (3) with the following parameters.

$$A = \begin{bmatrix} -1.2 & 0 \\ 0.8 & -1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -1 & 0.6 \\ -0.6 & -1 \end{bmatrix}, \qquad B_p = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \qquad B_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$C_q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad C_z = [1 \quad 0], \qquad \| \Delta A(t) \| \leq 0.2\alpha, \qquad \| \Delta A_1(t) \| \leq 0.03\alpha,$$

$$\Delta B_p(t) \leq 0.03\alpha, \qquad \phi \in \Phi\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1+\delta l \\ 3 \end{bmatrix} \right).$$

The uncertainty matrices $\Delta A(t)$, $\Delta A_1(t)$ and $\Delta B_p(t)$ can be represented with Eq. (4) where

$$D = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \qquad E_0 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \qquad E_1 = E_2 = \begin{bmatrix} 0.03 & 0 \\ 0 & 0.03 \end{bmatrix},$$

and $F^T(t)F(t) \leq I$, $F(t) \in \mathbf{R}^{2\times 2}$. This example is modified from the LSTD given in [8]. Again, loop transformation is applied so that LSTD with ϕ defined above is transformed to an equivalent LSTD with $\overline{\phi} \in \Phi(\mathbf{0}, \mathbf{1})$. Note that the new $\overline{L}$ and $\overline{M}$ are the zero and identity matrices of dimension 2×2, respectively.

Choosing $\delta l = 0.3$, $\alpha = 0.5$, and $\overline{h} = 0.5$, the minimal γ_∞ calculated using Theorem 2 is $\gamma_\infty = 2.4973$ with

$$P = \begin{bmatrix} 2.8524 & 0.7441 \\ 0.7441 & 6.3480 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 1.2187 & 1.5069 \\ 1.5069 & 1.8632 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 2.7965 & 3.8272 \\ 3.8272 & 5.2379 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 5.3964 & 8.2806 \\ 8.2806 & 12.7062 \end{bmatrix}, \qquad R_1 = \begin{bmatrix} 13.2874 & -2.3489 \\ -2.3489 & 48.5953 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 16.3210 & -1.7896 \\ -1.7896 & 45.8516 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 19.8893 & 0.0982 \\ 0.0982 & 44.0708 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} 3.2683 \times 10^{-7} & 0 \\ 0 & 0.4308 \end{bmatrix}, \qquad T = \begin{bmatrix} 5.1811 & 0 \\ 0 & 12.5544 \end{bmatrix},$$

$\gamma_\infty^2 = 6.2365$, and $\epsilon = 8.0454$ satisfying LMI (18). By Theorem 3, the minimal $\gamma_\infty = 13.7828$ is obtained with

$$P = \begin{bmatrix} 13.6848 & 2.1636 \\ 2.1636 & 36.1103 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 4.8453 & 12.3735 \\ 12.3735 & 31.5987 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 7.9113 & 19.8392 \\ 19.8392 & 49.7511 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 12.2948 & 30.4124 \\ 30.4124 & 75.2282 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 63.7547 & -36.6045 \\ -36.6045 & 319.0559 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 75.0087 & -28.1113 \\ -28.1113 & 298.0908 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 91.9836 & -14.9039 \\ -14.9039 & 267.9265 \end{bmatrix},$$

$\tau = 42.1115$, $\gamma_\infty^2 = 189.9651$, and $\epsilon = 35.9107$ satisfying LMI (21). Finally, the minimal $\gamma_\infty = 13.8281$ is computed using Theorem 4 when LMI (22) is satisfied by

$$P = \begin{bmatrix} 13.6697 & 1.9860 \\ 1.9860 & 35.6116 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 4.8005 & 12.2063 \\ 12.2063 & 31.0370 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 10.1062 & 25.0163 \\ 25.0163 & 61.9240 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} 10.5589 & -6.3718 \\ -6.3718 & 52.4851 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 14.1647 & -3.2270 \\ -3.2270 & 47.8418 \end{bmatrix}, \quad M_{11} = \begin{bmatrix} -30.3545 & 14.0202 \\ 30.6170 & -429.2920 \end{bmatrix},$$

$$M_{12} = \begin{bmatrix} 27.1573 & -50.2583 \\ 12.6749 & -1.4405 \end{bmatrix}, \quad M_{13} = \begin{bmatrix} 13.8097 & 25.0927 \\ -27.3479 & 400.3069 \end{bmatrix}, \quad M_{14} = \begin{bmatrix} -8.2095 & 6.4229 \\ -1.6478 & 22.6368 \end{bmatrix},$$

$$M_{21} = \begin{bmatrix} -21.6311 & 5.2550 \\ 62.8607 & 7.8385 \end{bmatrix}, \quad M_{22} = \begin{bmatrix} 33.2550 & -5.4683 \\ -9.9198 & 137.2605 \end{bmatrix}, \quad M_{23} = \begin{bmatrix} -16.6608 & -2.2491 \\ -55.1709 & -145.8085 \end{bmatrix},$$

$$M_{24} = \begin{bmatrix} 7.0427 & 3.5606 \\ -3.2502 & -1.1583 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 2.3083 \times 10^{-7} & 0 \\ 0 & 1.2448 \times 10^{-7} \end{bmatrix},$$

$\tau = 42.2465$, $\gamma_\infty^2 = 191.2156$, and $\epsilon = 36.0358$.

The plots of calculated γ_∞ for Ex. 3 are shown in Figs. 7–9. It is clearly seen that there is a large improvement between γ_∞ provided by Theorem 2 and those obtained from comparative criteria.

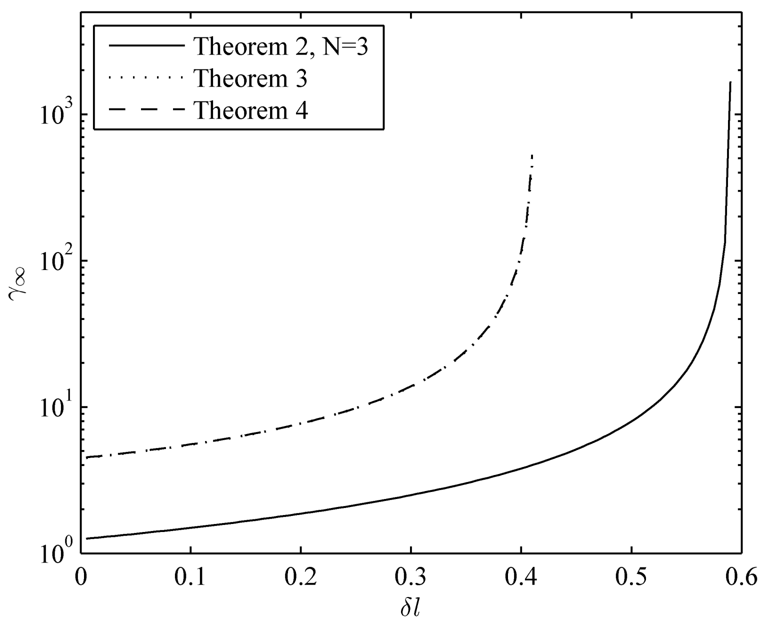

Fig. 7. Upper bounds of J_∞ for Ex. 3 when $\bar{h} = 0.5$ and $\alpha = 0.5$.

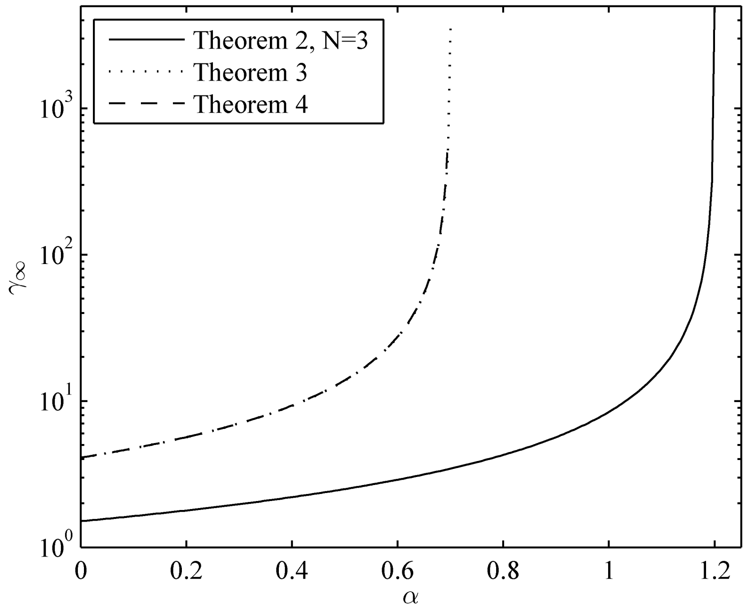

Fig. 8. Upper bounds of J_∞ for Ex. 3 when $\bar{h} = 0.5$ and $\delta l = 0.3$.

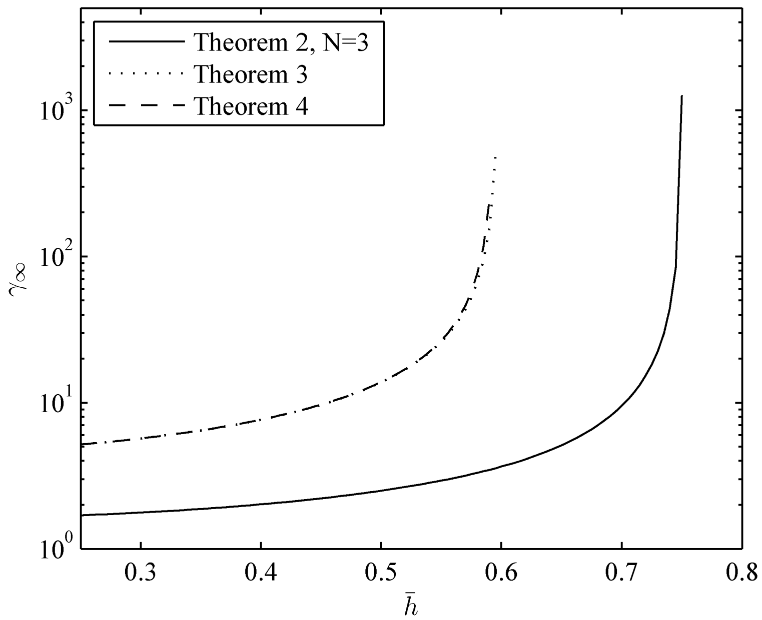

Fig. 9. Upper bounds of J_∞ for Ex. 3 when $\delta l = 0.3$ and $\alpha = 0.5$.

From the numerical results above, we observe that γ_∞ is increased when the sector-bounds, the uncertainties, or the bound of time delay is increased. The extension of [9] in Theorem 3 and the extension of [11] in Theorem 4, which can be viewed as a special case of our proposed criterion, always give the greater γ_∞ comparing to that obtained from our criterion. Therefore, we can conclude that the proposed criterion is less conservative than the comparative criteria.

5. Conclusions

In this paper, we present the worst-case H_∞ performance criteria for Lur'e systems with uncertain time-invariant delays. The delay partitioning technique is applied and the information of sector-bounded nonlinearities is incorporated into the LKF in terms of integral of nonlinearities. The sufficient condition to ensure the worst-case performance is derived using Jensen inequality and S-procedure. The performance criterion is formulated as a linear objective minimization problem over LMIs, which can be solved efficiently. In addition, the criterion for LSTD subject to norm-bounded uncertainties is developed by eliminating an uncertain matrix. Numerical examples show that the proposed criteria are less conservative than the comparative criteria, and can be served as an effective worst-case performance analysis for LSTD.

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