Engineering Journal issn: 0125-8281

Review

A Review of Chaos Control Strategies for Tri-trophic Food Chain Ecological Systems

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Abstract. The existence of chaos in ecological models is quite obvious due to the presence of nonlinear terms. Controlling chaotic phenomena in ecological systems remains a difficult task due to their unpredictability, and thus chaos control is one of the main objectives for constructing mathematical models in ecology today. Our aim in this paper is to review chaos control strategies for the tri-trophic food chain models by using various ecological factors. The factors include additional food, prey refuge, the Allee effect, the fear effect, and harvesting. We establish the essential conditions for the existence of ecologically feasible equilibrium points in the food chain ecological systems and their local stability. This paper provides a unified overview of recent research on the chaos control of ecological systems. The theoretical results suggest a way to control populations of species in ecological systems for fishing and pest management in farming. Numerical examples are performed to justify and compare the theoretical findings through phase portraits and bifurcation diagrams.

Keywords: Food chain model, chaos control strategies, ecological systems, additional food, prey refuge, Allee effect, fear effect, species harvesting.

ENGINEERING JOURNAL Volume 27 Issue 9
Received 8 February 2023
Accepted 19 September 2023
Published 30 September 2023
Online at https://engj.org/
DOI:10.4186/ej.2023.27.9.39

1. Introduction

The study of interactions between living things and their surroundings has recently received a lot of attention from theoretical ecologists since it is crucial in the formation of an ecosystem. Lotka and Volterra independently developed a pair of first-order ordinary differential equations to describe the process of interaction between two species in the mid-1920s. Following on from these pioneering works, researchers have demonstrated a strong interest in modeling and studying species interactions in both theoretical and experimental settings (see [1, 2] and references therein). To make the ecological model more realistic, a number of real elements have been incorporated, such as the Allee effect, prey refuges, stage structure, harvesting, toxic effects, and environmental variations. On the other hand, chaotic dynamics are quite obvious when nonlinear terms are present in ecological systems. Generally, predicting the future evolution of chaotic systems remains a difficult task due to its sensitivity to the initial condition and system parameters. Chaotic oscillations may tend toward possible population extinction at low densities. The authors in [3] stated that chaos and its control are a more significant and challenging phenomenon. However, forecasting future populations is much more important for maintaining a healthy ecosystem. Thus, the problem of chaos control in ecological models has become an important topic among theoretical ecologists in recent years.

Following Hastings and Powell's [4] work, several scientists developed numerous chaos control strategies in food chain models by incorporating various ecological components, (see [5–10]). For example, by incorporating an omnivore as the top predator, the chaotic behavior in the food chain model is eliminated, and periodic oscillation is obtained [6]. The authors in [5] introduced another predator that consumes only basal prey to control the chaos in the food chain model given in [4]. Gupta and Yadav [7] used a nonlinear harvesting strategy to keep the food chain model stable, where the harvesting of middle predators is a key part. Recently, the dynamics of the food chain model with an effect of cannibalism on the middle predator have been addressed in [8]. They concluded that chaotic dynamics in both temporal and spatiotemporal systems can be controlled by parameters of the cannibalism effect and intraspesific competition. Hossain et al. [9] have investigated the dynamics of a food chain model with vigilance effects, which reduce the growth rate of prey and middle predators. The authors demonstrated that both the vigilance of prey and the vigilance of middle-predator parameters are able to stabilize the system. On the other hand, excessive vigilance can harm the system and lead to the extinction of one or more populations.

The appropriate harvesting of prey populations helps stabilize a predator-prey-parasite system from chaotic dynamics [11]. In an eco-epidemiological model, increasing the predator mortality rate can reduce chaos [12]. Hossain et al. [13] have considered the effect of nanoparticles in an aquatic food chain model. They concluded that the effect of nanoparticles on prey growth can stabilize the system's chaotic oscillations. It has also been observed that excessive exposure to nanoparticles causes the extinction of one or more species. Additional food for zooplankton can play a significant role in maintaining the stability of the phytoplankton-zooplankton-fish (PZF) system [14]. Sajan et al. [15] considered the problem of chaos control in the PZF system through fear and additional food, where the system behaves chaotically when the cost of fear is low and then settles into a stable state as the cost of fear increases. Also, existing chaos at low levels of fear can be regulated through additional food. Dubey and Sasmal [16] studied a PZF system where fish produced fear and its carryover effects affected zooplankton growths. They noticed that for the midrange of the carryover effect parameters, the system exhibits chaotic behavior, and for lower and higher values, the system develops stable or periodic dynamics. The Allee effect in prey populations causes chaotic dynamics in the eco-epidemiological two-species model [17].

The Hasting-Powell (H-P) model assumes that prey, middle, and top predators are at the bottom, middle, and top trophic levels; see Fig. 1. It also assumes that middle predator growth is solely dependent on prey populations and that top predator growth is solely dependent on middle predators. The interaction between species in the H-P model is followed by the Holling type II function response. The H-P model exhibits chaotic behavior with respect to the half-saturation coefficient of prey. Chaos was found in this study, which suggests that chaotic dynamics are probably common in real food chain models [4]. In this connection, many theoretical ecologists are right now working on how to control chaos in ecological systems, but it is a challenging task due to their sensitivity to the system parameters. Recent works attempt to control the chaos in the H-P model by incorporating various ecological features. The new control methods should be easy to use, not involve chemicals, and not mess up the ecosystem. In the literature, the H-P model uses the Allee effect in prey growth [18, 19], supplying additional food to the predator [20, 21], the prey's refuge from avian predation [18, 22], the fear effect due to predation risk [23–25], species harvesting [22, 26, 27], vigilance behavior [9], gestation delay [28], and the nanoparticles effect [13] to keep chaos under control. Table 1 shows the different characteristics used in the H-P model (5) to deal with chaos.

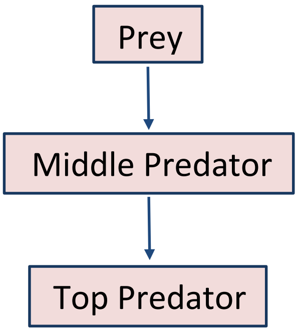

Fig. 1. The feeding relationship between the species in H-P food chain model.

Table 1. Summary of the ecological characteristics used in H-P model.

Model	Ecological characteristics used
[20, 21]	Additional Food to predators
[18, 19]	Allee effect
[18, 22]	Prey refuge
[23–25]	Fear due to predation risk
[22, 26, 27]	Harvesting of species
[9]	Vigilance behavior
[28]	Gestation delay
[13]	Nanoparticles effect

In this paper, our aim is to review non-chemical chaos control strategies that have received special attention in recent years for a chaotic H-P food chain model. The main motivation of this paper is to examine how ecological control strategies affect the dynamics of the H-P model in a unified framework. The focused control strategies consist of (i) providing additional food to middle and top predators; (ii) Allee effects on prey growth and prey refuge; (iii) fear due to predation risk; and (iv) species harvesting. We believe that this review will be useful to researchers working on the chaos control of ecological systems to gain an insightful and unified overview of recent research. Our theoretical findings also provide a way to protect and control populations of species in ecological systems, which is vitally applicable for fishing and pest management in farming.

The paper is organizaed as follows. We will concentrate on chaos control strategies in the H-P model. In the next section, we describe the nonlinear dynamic model of the food chain ecological systems. Section 3 investigates the additional food to predators. Section 4 addresses the Allee effect and the prey refuge. Then, the fear effect due to predation risk is analyzed in Section 5. Species harvesting is investigated in Section 6. In each section, we will derive the equilibrium points of the food chain model and their local stability conditions. Furthermore, we verify the effectiveness of chaos control method by performing numerical simulation in each section. Finally, in Section 7, we give the conclusions and suggest the future research.

2. Food Chain Ecological Systems

The utilization of supplementary food (non-prey food) is one of the fundamental approaches for biological control, including species conservation and pest management, and these foods fundamentally shape the life histories of many predator species. In a predator-prey system, the availability of sufficient supplementary food can significantly affect the dynamics of the system. In recent years, many biologists, experimentalists, and theoreticians have investigated the consequences of providing additional food to predators in predator-prey systems. Almost all predators will attempt to switch to another prey when their preferred prey is in low numbers, and they may also resort to scavenging or an herbivorous diet if possible. Many aspects of additional food were investigated in the literature; controlling chaos through the supply of additional food was one of them. This technique is one of the non-chemical methods, and it has no pollution factor or infection risk for the population system. In their works [20] and [21], Sahoo and Poria look at the H-P model with additional food supplies for the top predator alone and for both predators. The authors showed that the chaotic behavior of the proposed models can be controlled by both the quality and quantity of the additional food that is given. They also discovered that too much additional food supply has an impact on the ecosystem's structure.

The general H-P food chain model is given in the following form:

$$
\begin{aligned}
\frac{dU}{dT} &= RU\left(1-\frac{U}{K}\right)-\frac{C_1A_1UV}{B_1+U},\\
\frac{dV}{dT} &= \frac{A_1UV}{B_1+U}-D_1V-\frac{A_2VW}{B_2+V},\\
\frac{dW}{dT} &= \frac{C_2A_2VW}{B_2+V}-D_2W,
\end{aligned}
\tag{1}
$$

where U, V and W are the respective densities of prey, middle, and top predators. All the system parameters are assumed to be non-negative. R and K

are growth rate of prey and environmental supporting capacity. A_1 (or A_2) is maximum attack rate of middle (or top) predator. B_1 (or B_2) represents half-saturation coefficient of prey (or middle predator). C_1^{-1} (or C_2) indicates the conversion efficiencies of middle (or top) predator. D_1 (or D_2) are death rate of middle (or top) predator.

3. Additional Food to Predators

Model (1) is modified by introducing the supply of additional food to predators as follows:

$$\begin{aligned}\frac{dU}{dT} &= RU\left(1-\frac{U}{K}\right)-\frac{C_1A_1UV}{B_1+\alpha\mu A+U},\\ \frac{dV}{dT} &= \frac{A_1(U+\mu A)V}{B_1+\alpha\mu A+U}-D_1V-\frac{A_2VW}{B_2+\alpha\nu A+V},\\ \frac{dW}{dT} &= \frac{C_2A_2(V+\nu A)W}{B_2+\alpha\nu A+V}-D_2W,\end{aligned} \tag{2}$$

where μA and νA are effective additional food sources for middle and top predators, respectively, that are constantly supplied by either nature or an external agency. The parameter α describes the quality of additional food, which supplied to predators.

To minimize the complexity of model (2), we devise non-dimensional scheme as follows:

$$U = Ku,\ V = \frac{K}{C_1}v,\ W = \frac{C_2K}{C_1}w,\ T = \frac{1}{R}t \tag{3}$$

and let $\alpha_1 = \frac{A_1K}{RB_1}$, $\alpha_2 = \frac{A_2C_2K}{RB_2C_1}$, $\beta_1 = \frac{K}{B_1}$, $\beta_2 = \frac{K}{C_1B_2}$, $\delta_1 = \frac{D_1}{R}$, $\delta_2 = \frac{D_3}{R}$, $c = \frac{B_1}{K}$, $e = \frac{B_2C_1}{K}$, $\xi = \frac{\mu A}{B_1}$, $\eta = \frac{\nu A}{B_2}$. Then the system (2) becomes

$$\begin{aligned}\frac{du}{dt} &= u(1-u)-\frac{\alpha_1uv}{1+\alpha\xi+\beta_1u},\\ \frac{dv}{dt} &= \frac{\alpha_1(u+c\xi)v}{1+\alpha\xi+\beta_1u}-\frac{\alpha_2vw}{1+\alpha\eta+\beta_2v}-\delta_1v,\\ \frac{dw}{dt} &= \frac{\alpha_2(v+e\eta)w}{1+\alpha\eta+\beta_2v}-\delta_2w,\end{aligned} \tag{4}$$

subject to the initial conditions $0 < u_0 = u(0), v_0 = v(0), w_0 = w(0) < \infty$. If there is no supply of additional food (i.e., $\xi = \eta = 0$) in model (4), then the system is reduced to the H-P model as follows:

$$\begin{aligned}\frac{du}{dt} &= u(1-u)-\frac{\alpha_1uv}{1+\beta_1u},\\ \frac{dv}{dt} &= \frac{\alpha_1uv}{1+\beta_1u}-\frac{\alpha_2vw}{1+\beta_2v}-\delta_1v,\\ \frac{dw}{dt} &= \frac{\alpha_2vw}{1+\beta_2v}-\delta_2w.\end{aligned} \tag{5}$$

3.1. Equilibrium Points and their Stability

The system (4) have following equilibrium points:

1. The species free equilibrium $E_0 = (0,0,0)$ always exists.
2. The predators free equilibrium $E_1 = (1,0,0)$ always exists.
3. The top-predator free equilibrium point $E_2 = (u_2, v_2, 0)$, where

$$\begin{aligned}u_2 =& \frac{\delta_1(1+\alpha\xi)-\alpha_1c\xi}{\alpha_1-\beta_1\delta_1},\\ v_2 =& \frac{(1-u_2)(1+\alpha\xi+\beta_1u_2)}{\alpha_1}.\end{aligned}$$

   It is clear that E_2 exists only when $u_2 < 1$ and $\delta_1(1+\alpha\xi) > \alpha_1c\xi,\ \alpha_1 > \beta_1\delta_1$.

4. The coexistence equilibrium $E^* = (u^*, v^*, w^*)$, where

$$\begin{aligned}v^* &= \frac{\delta_2(1+\alpha\eta)-\alpha_2e\eta}{\alpha_2-\beta_2\delta_2},\\ w^* &= \frac{(1+\alpha\eta+\beta_2v^*)}{\alpha_2}\left(\frac{\alpha_1(u^*+c\xi)}{1+\alpha\xi+\beta_1u^*}-\delta_1\right),\end{aligned}$$

   and u^* is a positive root of

$$\begin{aligned}\beta_1(\alpha_2-\beta_2\delta_2)u^2+(\alpha_2-\beta_2\delta_2)(1+\alpha\xi-\beta_1)u&\\ +\alpha_1(\delta_2(1+\alpha)-\alpha_2e\eta)&=0.\end{aligned}$$

Note that the equilibrium E^* exists only when $\alpha_2 > \beta_2\delta_2$, $\delta_2(1+\alpha\eta) > \alpha_2e\eta$, and $\frac{\alpha_1(u^*+c\xi)}{1+\alpha\xi+\beta_1u^*} > \delta_1$.

To study the local stability of the equilibrium point, the Jacobian matrix of the system (4) at any point (u, v, w) is given by:

$$J(u,v,w) = \begin{bmatrix} j_{11} & j_{12} & 0\\ j_{21} & j_{22} & j_{23}\\ 0 & j_{32} & j_{33}\end{bmatrix}, \tag{6}$$

where

$$\begin{aligned}j_{11} =& 1-2u-\frac{\alpha_1v(1+\alpha\xi)}{(1+\alpha\xi+\beta_1u)^2},\\ j_{12} =& -\frac{\alpha_1u}{(1+\alpha\xi+\beta_1u)},\quad j_{21} = \frac{\alpha_1v(1+\alpha\xi-\beta_1c\xi)}{(1+\alpha\xi+\beta_1u)^2},\\ j_{22} =& \frac{\alpha_1(u+c\xi)}{1+\alpha\xi+\beta_1u}-\frac{\alpha_2w(1+\alpha\eta)}{(1+\alpha\eta+\beta_2v)^2}-\delta_1,\\ j_{23} =& -\frac{\alpha_2v}{1+\alpha\eta+\beta_2v},\quad j_{32} = \frac{\alpha_2w(1+\alpha\eta-\beta_2e\eta)}{(1+\alpha\eta+\beta_2v)^2},\\ j_{33} =& \frac{\alpha_2(v+e\eta)}{1+\alpha\eta+\beta_2v}-\delta_2.\end{aligned}$$

In the following theorem, we describe the local stability of model (4) at the equilibrium points.

Theorem 1. *For model (4):*

(i) The equilibrium E_0 is always unstable.

(ii) The equilibrium E_1 is stable if $\frac{\alpha_1(1+c\xi)}{1+\alpha\xi+\beta_1} < \delta_1$, and $\frac{\alpha_2 e\eta}{1+\alpha\eta} < \delta_2$.

(iii) If the equilibrium E_2 exists and $\alpha_1\beta_1 < (1+\alpha\xi+\beta_1 u_2)^2$ and $\frac{\alpha_2(v_2+e\eta)}{1+\alpha\eta+\beta_2 v_2} < \delta_2$, then E_2 is stable.

(iv) The equilibrium E^* exists and is locally asymptotically stable if $\Omega_1 > 0, \Omega_3 > 0$ and $\Omega_1\Omega_2 > \Omega_3$, where Ω_1, Ω_2 and Ω_3 are given in the proof.

Proof. (i) The eigenvalues of Jacobian matrix (6) at E_0 are $\lambda_1 = 1, \lambda_2 = -\delta_1$ and $\lambda_3 = -\delta_2$. Clearly, 1 is always positive, and $-\delta_1$ and $-\delta_2$ are always negative. So, E_0 is always unstable.

(ii) At the equilibrium E_1, the Jacobian matrix (6) have following equilibrium points $\lambda_1 = -1, \lambda_2 = \frac{\alpha_1(1+c\xi)}{1+\alpha\xi+\beta_1} - \delta_1$ and $\lambda_3 = \frac{\alpha_2 e\eta}{1+\alpha\eta} - \delta_2$. Clearly, -1 is always negative, and λ_2 and λ_3 are negative if $\frac{\alpha_1(1+c\xi)}{1+\alpha\xi+\beta_1} < \delta_1$ and $\frac{\alpha_2 e\eta}{1+\alpha\eta} < \delta_2$, which assures the local stability of E_1.

(iii) The eigenvalues of Jacobian matrix (6) associated with the equilibrium E_2 are $\lambda_1 = \bar{a}_{33}$ and $\lambda_{2,3} = \frac{\bar{a}_{11} \pm \sqrt{\bar{a}_{11}^2 + \bar{a}_{12}\bar{a}_{21}}}{2}$, where $\bar{a}_{mn}$ are obtained from j_{mn} replacing (u, v, w) by $(u_2, v_2, 0)$ in equation (6). So, the equilibrium E_2 is stable if $\alpha_1\beta_1 < (1+\alpha\xi+\beta_1 u_2)^2$ and $\frac{\alpha_2(v_2+e\eta)}{1+\alpha\eta+\beta_2 v_2} < \delta_2$.

(iv) The characteristic equation of (6) corresponding to E^* is

$$\lambda^3 + \Omega_1\lambda^2 + \Omega_2\lambda + \Omega_3 = 0,$$

where $\Omega_1 = -a_{11} - a_{22}$, $\Omega_2 = a_{11}a_{22} - a_{12}a_{21} - a_{32}a_{23}$, $\Omega_3 = a_{11}a_{23}a_{32}$. Here, a_{mn} are obtained from j_{mn} replacing (u, v, w) by (u^*, v^*, w^*) in equation (6). Make use of Routh-Hurwitz (R-H) criterion, the equilibrium point E^* is locally asymptotically stable if $\Omega_1 > 0, \Omega_3 > 0$ and $\Omega_1\Omega_2 > \Omega_3$ holds. □

3.2. Numerical Simulation

The phase portraits and bifurcation diagrams for the models discussed in this paper are drawn using the initial populations $(0.75, 0.15, 10)$. The middle predator's bifurcation diagram is only used because all three species in the system exhibit identical bifurcation diagrams. All of the system parameters of the model (6) are chosen from the reference [4]. First, the numerical simulations of the model (4) without supply of additional food (i.e. $\xi = \eta = 0$) is carried and observed that the dynamics of the system switches between stable focus, periodic oscillations and chaos for varying the parameter β_1, while fixing other parameters as $\alpha_1 = 5, \alpha_2 = 0.1$, $\beta_2 = 2, \delta_1 = 0.4, \delta_2 = 0.01$. In Fig. 2, we plot phase portrait of the model (5) for $\beta 1 = 3.0$ and other parameters given the same as above, which show chaotic oscillation. Figure 3 represents the bifurcation diagram and fluctuation of the largest Lyapunov exponent (LLE) of the model (5) with respect to the parameter β_1. The system shows that stable focus for $\beta_1 < 2.1$, limit cycle behavior for $2.1 \leq \beta_1 < 2.3$, period-doubling oscillations for $2.3 \leq \beta_1 < 2.38$ and chaotic oscillations for $\beta_1 \geq 2.38$. The positive values of the LLE ensure the existence of chaotic behaviours in the system (5); see Fig. 3b.

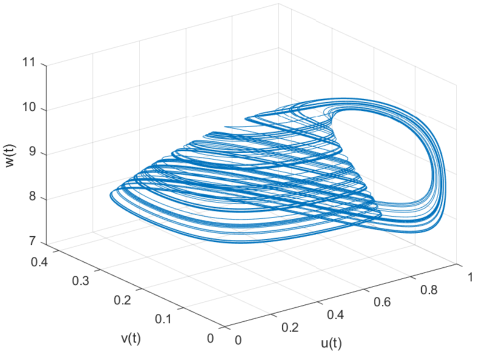

Fig. 2. The phase portrait of the model (5) for $\beta_1 = 3.0$ with $\alpha_1 = 5, \alpha_2 = 0.1$, $\beta_2 = 2, \delta_1 = 0.4$, $\delta_2 = 0.01$.

Now we investigate the influence supplying additional food on model (4) by varying respective parameters and fixing other parameters as $\alpha_1 = 5$, $\alpha_2 = 0.1$, $\beta_1 = 3.0$, $\beta_2 = 2, \delta_1 = 0.4, \delta_2 = 0.01$, $\xi = 0.1, \eta = 0.1$, $c = 0.95, e = 0.85$. Figure 4 depicts the bifurcation diagram and fluctuation of the LLE of the system (4) with respect to the quality of additional food parameter α in the interval $[0, 12]$. It is noticeable from Fig. 4a that for lower values of quality of additional food α, the system shows chaotic or higher periodic oscillations. Then with the increase of values of α the system enters into stable state from chaos via periodic oscillations. The positive LLE values shown in Fig. 4b confirm the chaotic dynamics that occur in Fig. 4a. Also, it is observed that any choice of quantity of additional food ξ and η lead to negative effects in the ecosystem, that is population goes to extinction stage. With a proper choice of supplying additional food parameters (both quality and quantity) to predators one can control system dynamics.

4. Allee Effect and Prey Refuge

The Allee effect, a biological phenomenon, demonstrates a connection between population size

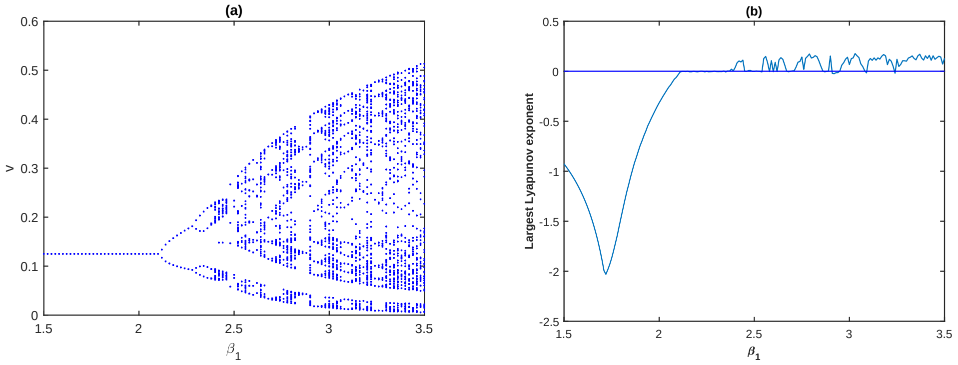

Fig. 3. For system (5): (a) a bifurcation diagram of the middle-predator with respect to parameter β_1; (b) fluctuation of the LLE with respect to β_1.

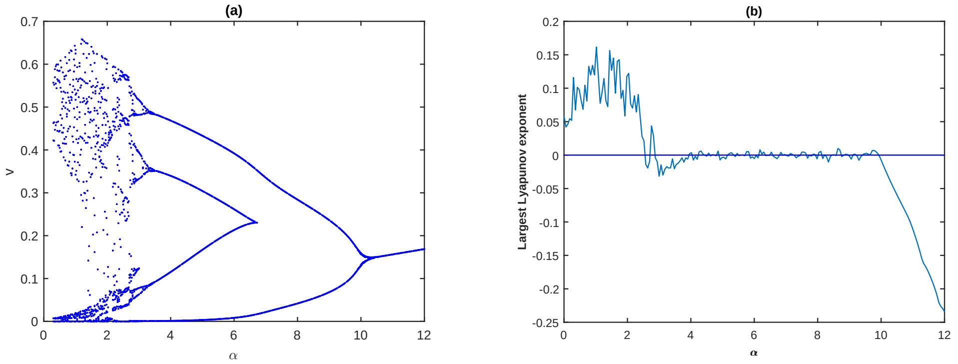

Fig. 4. For system (4): (a) a bifurcation diagram of the middle-predator with respect to parameter α; (b) fluctuation of the LLE with respect to α.

and their per capita growth rate when the density of the population is low. W.C. Allee first introduced the concept of the Allee effect in his pioneering works [29, 30]. The Allee effect among a natural population is commonly caused by mate restriction, cooperative defense, predator satiation, cooperative feeding, dispersal, habitat modification, and so on. There are two types of Allee effects: strong Allee effects and weak Allee effects. The strong Allee effect takes place when population densities fall below a critical threshold, which causes negative per capita growth rates. In contrast, populations with a weak Allee effect have lower per capita growth rates at low densities, but they never reach negative per capita growth rates and have no critical threshold to exceed. Numerous natural species have been studied for experiential evidence of the Allee effect, including insects [31], marine invertebrates [32], mammals, and birds [33]. On the other hand, a refugium is a place where species can persist due to some special environmental features. In order to maintain the biodiversity of the bird populations in the Amazonian river basin, Haffer was the one who first proposed the concept of refugia. Numerous analytical and experimental studies have been conducted, and it has been noticed that prey refuges maintain the stability of the predator-prey model (see [34]). Authors in [18] have employed the Allee effect and prey refuge strategies to avoid chaotic dynamics in the H-P model. Recently, Mandal et al. [19] used the additive Allee effect concept in the H-P model and demonstrated that chaotic oscillations could be controlled by varying the severity of the Allee effect parameter.

The Allee effect on prey growth term and prey refuge are taken into account in the H-P model, and the non-dimensional version of the modified model takes the following form:

$$
\begin{aligned}
\frac{du}{dt} &= u(1-u)\left(\frac{u}{u+\theta}\right) - \frac{\alpha_1(1-m)uv}{1+\beta_1(1-m)u},\\
\frac{dv}{dt} &= \frac{\alpha_1(1-m)uv}{1+\beta_1(1-m)u} - \frac{\alpha_2 vw}{1+\beta_2 v} - \delta_1 v,\\
\frac{dw}{dt} &= \frac{\alpha_2 vw}{1+\beta_2 v} - \delta_2 w,
\end{aligned}
\tag{7}
$$

subject to the initial conditions $0 < u_0 = u(0), v_0 = v(0), w_0 = w(0) < \infty$. The term $\frac{u}{u+\theta}$ describes the Allee effect on prey's growth. $m \in [0,1)$ is refuge coefficient of prey. It is assumed that the $m \in [0,1)$ portion of the prey always avoids middle predators, while the remaining $(1-m)$ portion only interacts with middle predators.

4.1. Equilibrium Points and their Stability

The model (4) has four feasible equilibrium points.

1. The species free equilibrium $E_0 = (0, 0, 0)$ always exists.
2. The predators free equilibrium $E_1 = (1, 0, 0)$ always exists.
3. The top-predator free equilibrium point $E_2 = (u_2, v_2, 0)$, where
$$u_2 = \frac{\delta_1}{(\alpha_1 - \beta_1\delta_1)(1-m)},$$
$$v_2 = \frac{u_2^2(1-u_2)}{\delta_2(u_2+\theta)}.$$
It is clear that E_2 exists only when $u_2 < 1$ and $\alpha_1 > \beta_1\delta_1$.
4. The coexistence equilibrium $E^* = (u^*, v^*, w^*)$, where
$$v^* = \frac{\delta_2}{\alpha_2 - \beta_2\delta_2},$$
$$w^* = \frac{1}{\delta_2}\left(\frac{u^{*2}(1-u^*)}{u^*+\theta} - \delta_1 v^*\right),$$
and u^* is a positive root of cubic equation
$$u^3 - \left(1 - \frac{1}{\beta_1(1-m)}\right)u^2 + \frac{\alpha_1 v^*}{\beta_1}u + \frac{\alpha_1\theta v^*}{\beta_1} = 0.$$

The equilibrium E^* exists only when $\alpha_2 > \beta_2\delta_2$, $\beta_1(1-m) > 1$, and $u^{*2}(1-u^*) > \delta_1(u^*+\theta)v^*$.

The Jacobian matrix of the system (7) at any point (u, v, w) is calculated and given by:

$$J(u,v,w) = \begin{bmatrix} j_{11} & j_{12} & 0 \\ j_{21} & j_{22} & j_{23} \\ 0 & j_{32} & j_{33} \end{bmatrix}, \tag{8}$$

where

$$j_{11} = \frac{u\left(2\theta - 2u^2 - 3\theta u + u\right)}{(\theta+u)^2} - \frac{\alpha_1(1-m)v}{(1+\beta_1(1-m)u)^2},$$
$$j_{12} = -\frac{\alpha_1(1-m)u}{(1+\beta_1(1-m)u)},$$
$$j_{21} = \frac{\alpha_1(1-m)v}{(1+\beta_1(1-m)u)^2},$$
$$j_{22} = \frac{\alpha_1(1-m)u}{1+\beta_1(1-m)u} - \frac{\alpha_2 w}{(1+\beta_2 v)^2} - \delta_1,$$
$$j_{23} = -\frac{\alpha_2 v}{1+\beta_2 v}, \; j_{32} = \frac{\alpha_2 w}{(1+\beta_2 v)^2},$$
$$j_{33} = \frac{\alpha_2 v}{1+\beta_2 v} - \delta_2.$$

The following theorem derives the local stability of the equilibrium point from Jacobian matrix (8).

Theorem 2. *For model (7):*

(i) The equilibrium E_0 is non-hyperbolic equilibrium point having stable manifold in $y-z$ plane.

(ii) The equilibrium E_1 is always stable.

(iii) If the equilibrium E_2 exists and $\alpha_2 v_2 < \delta_2(1+\beta_2 v_2)$ and $\frac{\alpha_1\beta_1(1-m)^2 v_2}{(1+\beta_1(1-m)u_2)^2} + \frac{\theta(\theta+1)}{(u_2+\theta)^2} < 1$, then E_2 is stable.

(iv) The equilibrium E^* exists and is locally asymptotically stable if $\Omega_1 > 0, \Omega_3 > 0$ and $\Omega_1\Omega_2 > \Omega_3$, where Ω_1, Ω_2 and Ω_3 are given in the proof.

Proof. (i) The eigenvalues of Jacobian matrix (8) at E_0 are $\lambda_1 = 0, \lambda_2 = -\delta_1$ and $\lambda_3 = -\delta_2$. Hence, E_0 is non hyperbolic equilibrium point and have stable manifold in $y-z$ plane.

(ii) At the equilibrium E_1, the Jacobian matrix (8) have following equilibrium points $\lambda_1 = -\frac{1}{1+\theta}, \lambda_2 = -\delta_1$ and $\lambda_3 = -\delta_2$. Clearly, all the eigenvalues are always negative, which implies E_2 is stable.

(iii) The eigenvalues of Jacobian matrix (8) associated with the equilibrium E_2 are $\lambda_1 = \bar{b}_{33}$ and $\lambda_{2,3} = \frac{\bar{b}_{11} \pm \sqrt{\bar{b}_{11}^2 + \bar{b}_{12}\bar{b}_{21}}}{2}$, where $\bar{b}_{mn}$ are obtained from j_{mn} replacing (u, v, w) by $(u_2, v_2, 0)$ in equation (8). So, the equilibrium E_2 is stable if $\alpha_1\beta_1 < (1 + \alpha\xi + \beta_1 u_2)^2$ and $\frac{\alpha_2(v_2+e\eta)}{1+\alpha\eta+\beta_2 v_2} < \delta_2$.

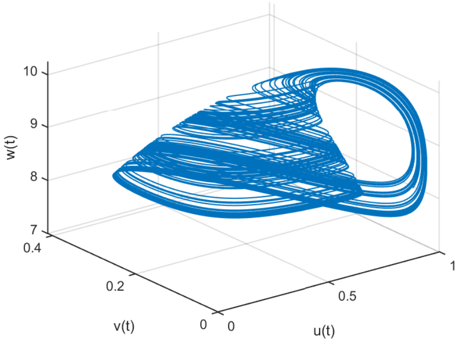

Fig. 5. The phase portrait of the model (7) for the set of parameters given in Fig. 2 with $m = 0.04$ and $\theta = 0.01$.

(iv) The characteristic equation of (6) corresponding to E^* is

$$\lambda^3 + \Omega_1\lambda^2 + \Omega_2\lambda + \Omega_3 = 0,$$

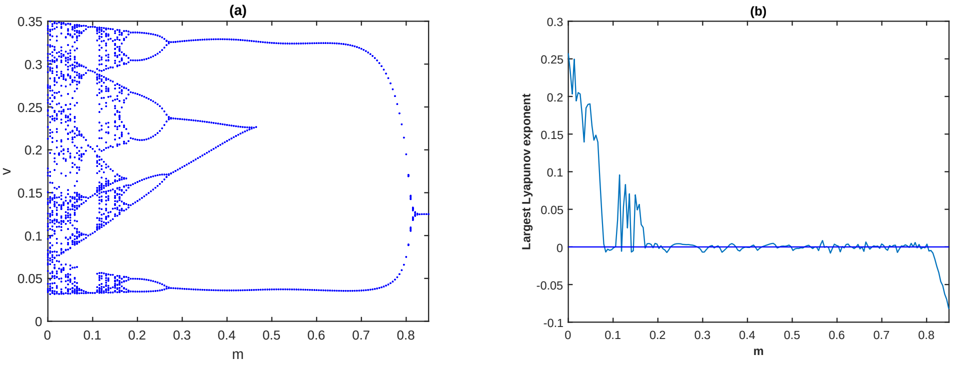

Fig. 6. For system (7) with $\theta = 0.05$: (a) a bifurcation diagram of the middle-predator with respect to parameter m; (b) fluctuation of the LLE with respect to m.

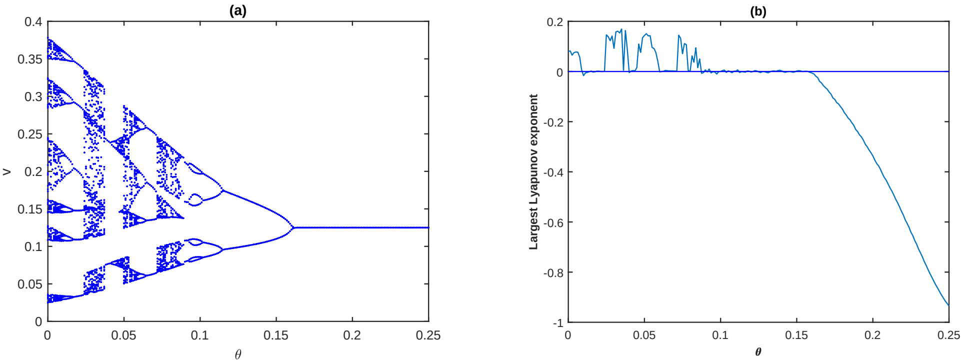

Fig. 7. For system (7) with $m = 0.04$: (a) a bifurcation diagram of the middle-predator with respect to parameter θ; (b) fluctuation of the LLE with respect to θ.

where $\Omega_1 = -b_{11} - b_{22}$, $\Omega_2 = b_{11}b_{22} - b_{12}b_{21} - b_{32}b_{23}$, $\Omega_3 = b_{11}b_{23}b_{32}$. Here, b_{mn} are obtained from j_{mn} replacing (u, v, w) by (u^*, v^*, w^*) in equation (8). By utilizing R-H criterion, the equilibrium point E^* is locally asymptotically stable if $\Omega_1 > 0, \Omega_3 > 0$ and $\Omega_1\Omega_2 > \Omega_3$ holds. □

4.2. Numerical Simulation

The system (7) exhibits chaotic oscillation for the set of parameters given in Fig. 2 with $m = 0.04$ and $\theta = 0.05$ so that the effectiveness of the Allee effect and predator refuge can be investigated, see Fig. 5. In Fig. 6, we draw a bifurcation diagram and fluctuation of the LLE to examine the behavior of system (7) with variations of m while keeping other parameters fixed. From Fig. 6a, it is noticed that for lower values of prey refuge, the system behaves chaotically, then is controlled and becomes stable as prey refuge increases. Additionally, we have seen that both predators become extinct at a threshold of $m = 0.88$. The system becomes stable from its chaotic dynamics through a period-halving oscillations with an increase in Allee effect parameter θ, see Fig. 7a. The chaotic nature of the system (7) with regard to the parameters m and θ is confirmed by the positive LLE values (see Figs. 6b. and Fig. 7b). From Figs. 6 and 7, it is clear that the Allee effect and prey refuge parameters play an important role in regulating the chaotic dynamics of a three-species food chain model.

5. Fear Effect

Predators may affect the population of their prey indirectly (through fear of predation) or directly (through killing). Fear has the potential to change the demographics of the prey and can have a substantial impact on the dynamics of the predator-prey system. To reduce the risk of predation, prey always try to shift their usual habitat to a safe location [35]. Based on the numerous experimental results on the fear effect, Wang et al. [36] mathematically examined the effect of fear on prey's growth and found that the cost of fear had a significant impact on the dynamics of predator-prey interactions. Since then, a number of studies on the effect of fear in predator-prey interactions have been reported [10,23] and the references therein. The authors of [23] investigated the dynamics of a food chain model with the fear effect, in which the growth of prey and middle predators shrinks due to the fear of middle predators and top predators, respectively. They showed that a suitable amount of fear was able to regularize the system from chaotic oscillation to stable focus through period-halving bifurcation. Furthermore, if the cost of fear for middle predators rises, the top predator may enter an extinction stage. Shi and Hu [25] considered and analyzed the three-species food chain model with fear, where the fear of predation risk is not instant but rather has some time lags on it. The authors in [24] discuss the three-species food chain model, in which fear of the top predator and middle predator foraging decreases. The dynamics of a food chain model with a Beddington-DeAngelis functional response have been examined by Debnath et al. [10]. They found that not only fear effects but also mutual interference parameters among species can control the system dynamics. Thus, fear is introduced in the food chain model (1), and the model then takes the following form:

$$\begin{aligned}\frac{dU}{dT} &= \frac{RU}{1+F_1V}\left(1-\frac{U}{K}\right)-\frac{C_1A_1UV}{B_1+U},\\ \frac{dV}{dT} &= \frac{A_1UV}{B_1+U}\frac{1}{1+F_2W}-D_1V-\frac{A_2VW}{B_2+V},\\ \frac{dW}{dT} &= \frac{C_2A_2VW}{B_2+V}-D_2W,\end{aligned}\tag{9}$$

in which F_1 and F_2 respectively represent the intensity of fear in prey and middle predator population. Model (9) assumes that due to fear of the top-predator, the growth rate of the middle-predator reduces, and due to fear of the middle-predator, the growth rate of prey also reduces. The fear functions $\Phi(K_1,V)=\frac{1}{1+F_1V}$ and $\Psi(K_2,W)=\frac{1}{1+F_2W}$ satisfies the following characteristics:

- $\Phi(0,V)=1$ and $\Psi(0,W)=1$,
- $\Phi(F_1,0)=1$ and $\Psi(F_2,0)=1$,
- $\lim_{F_1\to\infty}\Phi=0$ and $\lim_{F_2\to\infty}\Phi=0$,
- $\lim_{V\to\infty}\Phi=0$ and $\lim_{W\to\infty}\Phi=0$,
- $\frac{\partial\Phi}{\partial F_1}<0,\ \frac{\partial\Phi}{\partial V}<0$ and $\frac{\partial\Psi}{\partial F_2}<0,\ \frac{\partial\Psi}{\partial W}<0$.

Using non-dimensional scheme as given in (3) and then model (9) becomes

$$\begin{aligned}\frac{du}{dt} &= \frac{u(1-u)}{1+f_1v}-\frac{\alpha_1uv}{1+\beta_1u},\\ \frac{dv}{dt} &= \frac{\alpha_1uv}{(1+\beta_1u)(1+f_2w)}-\frac{\alpha_2vw}{1+\beta_2v}-\delta_1v,\\ \frac{dw}{dt} &= \frac{\alpha_2vw}{1+\beta_2v}-\delta_2w,\end{aligned}\tag{10}$$

subject to the initial conditions $0<u_0=u(0), v_0=v(0), w_0=w(0)<\infty$. Here, $f_1=\frac{F_1KC_2}{C_1}$, $f_2=\frac{F_2KC_2}{C_1}$ and the remaining parameters are same as in model (5).

5.1. Equilibrium Points and their Stability

The system (10) have following equilibrium points:

1. The species free equilibrium $E_0=(0,0,0)$ always exists.
2. The predators free equilibrium $E_1=(1,0,0)$ always exists.
3. The top-predator free equilibrium point $E_2=(u_2,v_2,0)$, where

   $$\begin{aligned}u_2 &= \frac{\delta_1}{(\alpha_1-\beta_1\delta_1)},\\ v_2 &= \frac{-\alpha_1+\sqrt{\alpha_1^2+4\alpha_1f_1(1-u_2)(1+\beta_1u_2)}}{2\alpha_1f_1}.\end{aligned}$$

   It is clear that E_2 exists only when $u_2<1$ and $\alpha_1>\beta_1\delta_1$.
4. The coexistence equilibrium $E^*=(u^*,v^*,w^*)$, where

   $$u^*=\frac{\beta_1-1+\sqrt{(1-\beta_1)^2-4\beta_1\epsilon}}{2\beta_1},$$
   $$\epsilon=\alpha_1v^*+\alpha_1f_1v^{*2}-1,\ v^*=\frac{\delta_2}{\alpha_2-\beta_2\delta_2},$$

   and w^* is the positive root of the following equation

   $$\frac{\alpha_2f_2}{1+\beta_2v^*}w^2+\left[\frac{\alpha_2}{1+\beta_2v^*}+f_2\delta_1\right]w-\left[\frac{\alpha_1u^*}{1+\beta_1u^*}-\delta_1\right]=0.$$

Note that E^* exists only when $\alpha_2>\beta_2\delta_2$, $u^*>\frac{\delta_1}{\alpha_1-\beta_1\delta_1}$, and $\alpha_1v^*(1+f_1v^*)$.

For the purpose of investigating the local stability of the equilibrium point, the Jacobian matrix of the system (4) at any point (u,v,w) is evaluated and given by:

$$J(u,v,w)=\begin{bmatrix} j_{11} & j_{12} & 0\\ j_{21} & j_{22} & j_{23}\\ 0 & j_{32} & j_{33}\end{bmatrix},\tag{11}$$

where

$$\begin{aligned}j_{11} &= \frac{1-2u}{1+f_1v}-\frac{\alpha_1v}{(1+\beta_1u)^2},\\ j_{12} &= -\frac{f_1u(1-u)}{(1+f_1v)^2}-\frac{\alpha_1u}{1+\beta_1u},\end{aligned}$$

$$\begin{aligned}
j_{21} =& \frac{\alpha_1 v}{(1+f_2 w)(1+\beta_1 u)^2}, \\
j_{22} =& \frac{\alpha_1 u}{(1+\beta_1 u)(1+f_2 w)} - \frac{\alpha_2 w}{(1+\beta_2 v)^2} - \delta_1, \\
j_{23} =& -\frac{\alpha_1 f_2 u v}{(1+f_2 w)^2(1+\beta_1 u)} - \frac{\alpha_2 v}{1+\beta_2 v}, \\
j_{32} =& \frac{\alpha_2 w}{(1+\beta_2 v)^2}, \quad j_{33} = \frac{\alpha_2 v}{1+\beta_2 v} - \delta_2.
\end{aligned}$$

In the next theorem, we give the sufficient conditions for local stability of the equilibrium points of model (10).

Theorem 3. *For model (10):*

(i) The equilibrium E_0 is always unstable.

(ii) The equilibrium E_1 is stable if $\alpha_1 < \delta_1(1+\beta_1)$.

(iii) If the equilibrium E_2 exists and $u_2 > \frac{\beta_1 - 1}{2\beta_1}$ and $v_2 < \frac{\delta_2}{\alpha_2 - \beta_2\delta_2}$, then E_2 is stable.

(iv) The equilibrium E^* exists and is locally asymptotically stable if $\Omega_1 > 0, \Omega_3 > 0$ and $\Omega_1\Omega_2 > \Omega_3$, where Ω_1, Ω_2 and Ω_3 are given in the proof.

Proof. (i) The eigenvalues of Jacobian matrix (11) at E_0 are $\lambda_1 = 1, \lambda_2 = -\delta_1$ and $\lambda_3 = -\delta_2$. Clearly, 1 is always positive, and $-\delta_1$ and $-\delta_2$ are always negative. So, E_0 is always unstable.

(ii) At the equilibrium E_1, the Jacobian matrix (11) have following equilibrium points $\lambda_1 = -1, \lambda_2 = \frac{\alpha_1}{1+\beta_1} - \delta_1$ and $\lambda_3 = -\delta_2$. Clearly, -1 and $-\delta_2$ are always negative, and λ_3 is negative if $\frac{\alpha_1}{1+\beta_1} < \delta_1$, which assures the local stability of E_1.

(iii) The eigenvalues of Jacobian matrix (11) associated with the equilibrium E_2 are $\lambda_1 = \bar{c}_{33}$ and $\lambda_{2,3} = \frac{\bar{c}_{11} \pm \sqrt{\bar{c}_{11}^2 + \bar{c}_{12}\bar{c}_{21}}}{2}$, where $\bar{c}_{mn}$ are obtained from j_{mn} replacing (u, v, w) by $(u_2, v_2, 0)$ in equation (11). So, the equilibrium E_2 is stable if $u_2 > \frac{\beta_1 - 1}{2\beta_1}$ and $v_2 < \frac{\delta_2}{\alpha_2 - \beta_2\delta_2}$.

(iv) The characteristic equation of (11) corresponding to E^* is

$$\lambda^3 + \Omega_1\lambda^2 + \Omega_2\lambda + \Omega_3 = 0,$$

where $\Omega_1 = -c_{11} - c_{22}$, $\Omega_2 = c_{11}c_{22} - c_{12}c_{21} - c_{32}c_{23}$, $\Omega_3 = c_{11}c_{23}c_{32}$. Here, c_{mn} are obtained from j_{mn} replacing (u, v, w) by (u^*, v^*, w^*) in equation (11). Make use of Routh-Hurwitz (R-H) criterion, the equilibrium point E^* is locally asymptotically stable $\Omega_1 > 0, \Omega_3 > 0$ and if $\Omega_1\Omega_2 > \Omega_3$ holds. □

5.2. Numerical Simulations

To investigate the effect of fear, we fix the system parameters as $\alpha_1 = 5$, $\alpha_2 = 0.1$, $\beta_1 = 3.0$, $\beta_2 = 2, \delta_1 = 0.4, \delta_2 = 0.01$, so that the model (10) exhibits chaotic oscillations in the absence of fear ($f_1 = f_2 = 0$), see Fig. 2. In the absence of top-predator fear ($f_2 = 0$), we first examine the impact of middle-predator fear on the growth rate of the basic prey. With the increase of f_1, the system becomes into stable state from its chaotic dynamics via period-halving bifurcation, see Fig. 8a. We noticed that the system shows chaotic oscillations in $f_1 < 1.05$; period-doubling oscillations in $1.05 \leq f_1 < 1.31$; limit cycle oscillations in $1.31 \leq f_1 < 1.80$, and stable behaviour in $f_1 \geq 1.80$. Also, we have seen that above a threshold value of $f_1 = 2.6$, the top predator goes extinct, whereas prey and the middle predator show oscillatory behavior. Similarly, we investigate that fear of the top-predator shrinks the growth rate of middle predator while the fear of middle-predator on prey is absent ($f_1 = 0$). The system becomes stable from its chaotic dynamics through a period-halving oscillations with an increase in fear parameter f_2, see Fig. 9a. The system shows chaotic dynamics in $f_2 < 0.03$; period doubling oscillations in the interval $0.03 \leq f_2 < 0.041$; periodic oscillations for $0.041 \leq f_2 < 0.29$ and stable focus for $f_2 \geq 0.29$. The positive LLE values in Figs. 8b and 9b, respectively, confirm the existence of chaotic dynamics in the system (10) depicted in Figs. 8a and 9a. In the presence of both fear parameters, we draw a bifurcation diagram in the $f_1 - f_2$ parametric plane in Fig. 10. We observe that, the system (10) exhibits chaotic behavior for lower values of fear parameter and then system becomes stable for higher values. From an ecological viewpoint, the growth rate of prey and middle predators reduces as the fear factors f_1 and f_2 increase, which results in a considerable reduction in the number of prey and middle predators that are consumed by middle and top predators. As a result of the discussion above, it can be concluded that both fear parameters can control the coexistence of all three species with other fixed system parameters.

6. Species Harvesting

Species harvesting has a significant impact on population dynamics. The intensity of this impact depends on the characteristics of the harvested strategy, which in turn may range from a sudden reduction to the total preservation of a population. The exploitation of natural resources and the harvesting of population species are commonly seen in fisheries, forestry, and wildlife management. The impact of harvesting on a natural population is one of the most important topics in population ecology. Inspired by this fact, the harvesting of pop-

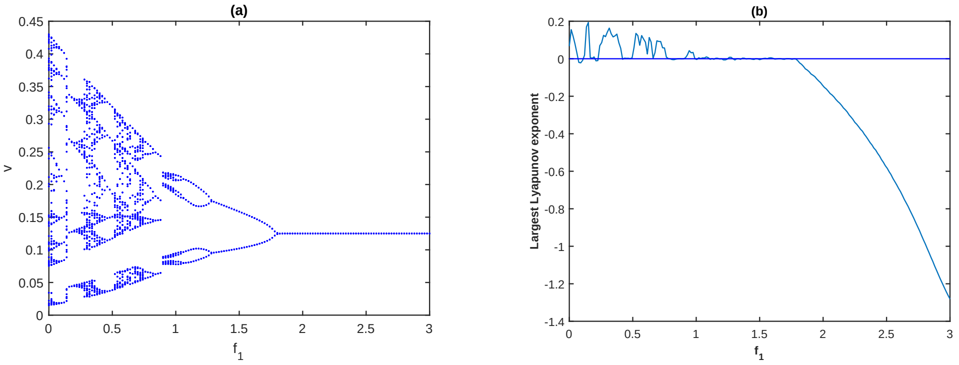

Fig. 8. For system (10) with $f_2 = 0$: (a) a bifurcation diagram of the middle-predator with respect to parameter f_1; (b) fluctuation of the LLE with respect to f_1.

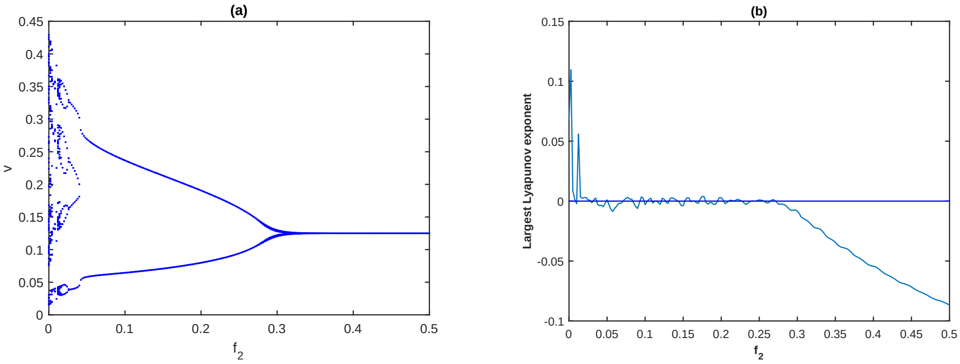

Fig. 9. For system (10) with $f_1 = 0$: (a) a bifurcation diagram of the middle-predator with respect to parameter f_2; (b) fluctuation of the LLE with respect to f_2.

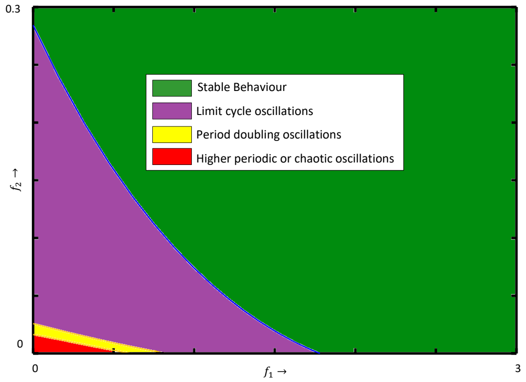

Fig. 10. The two parameter bifurcation diagram of model (10) in $f_1 - f_2$ parametric space; where the system exhibits stable behavior, limit cycle, period-doubling, and chaotic oscillations in green, purple, yellow, and red regions, respectively.

ulations has been incorporated into modeling the prey-predator system, and some interesting results with harvesting have been presented in [26, 37] and references therein. Harvesting can lead to a reduction in the size of the population due to hunting or capturing species. There are different types of harvesting strategies used in the literature, including constant-yield harvesting, linear harvesting, and nonlinear harvesting. Nath and Das [27] have investigated the impact of a linear harvesting strategy in a three-species food chain model, where they assumed that all three-species are harvested. They concluded that harvesting parameters can control the chaotic dynamics of the system. Raw and Sarangi [22] have dealt with the dynamics of the three species fishery model with nonlinear harvesting and prey refuge. By employing the linear harvesting method in the model

(1), the model is described as follows:

$$\begin{aligned}\frac{dU}{dT} &= RU\left(1-\frac{U}{K}\right)-\frac{C_1A_1UV}{B_1+U}-H_1U,\\ \frac{dV}{dT} &= \frac{A_1UV}{B_1+U}-D_1V-\frac{A_2VW}{B_2+V}-H_2V,\\ \frac{dW}{dT} &= \frac{C_2A_2VW}{B_2+V}-D_2W-H_3W,\end{aligned} \tag{12}$$

where H_1, H_2 and H_3 are the harvesting rate of the prey, middle-predator and top-predator populations, respectively. Here we assume that $H_i \in [0,1),\ i = 1,2,3$. The non-dimensional version of model (12) becomes:

$$\begin{aligned}\frac{du}{dt} &= u(1-u)-\frac{\alpha_1uv}{1+\beta_1u}-h_1u,\\ \frac{dv}{dt} &= \frac{\alpha_1uv}{(1+\beta_1u)}-\frac{\alpha_2vw}{1+\beta_2v}-\delta_1v-h_2v,\\ \frac{dw}{dt} &= \frac{\alpha_2vw}{1+\beta_2v}-\delta_2w-h_3w,\end{aligned} \tag{13}$$

subject to the initial conditions $0 < u_0 = u(0), v_0 = v(0), w_0 = w(0) < \infty$. Here $h_1 = \frac{H_1}{R},\ h_2 = \frac{H_2}{R}, h_3 = \frac{H_3}{R}$ and the remaining parameters are same as in model (5). Note that the harvesting parameters h_1, h_2, h_3 can not exceed their maximum value 1 because harvesting more than 100% of any individuals is not at all possible.

6.1. Equilibrium Points and their Stability

The model (10) possesses the equilibrium points as follows:

1. The species free equilibrium $E_0 = (0,0,0)$ always exists.
2. The predators free equilibrium $E_1 = (1-h_1, 0, 0)$ exists only when $h_1 < 1$.
3. The top-predator free equilibrium point $E_2 = (u_2, v_2, 0)$, where
$$\begin{aligned}u_2 =& \frac{\delta_1+h_2}{(\alpha_1-\beta_1\delta_1-\beta_1h_2)},\\ v_2 =& \frac{(1-h_1)(\alpha_1-\beta_1\delta_1-\beta_1h_2)-(\delta_1+h_2)}{(\alpha_1-\beta_1\delta_1-\beta_1h_2)^2}.\end{aligned}$$
It is clear that E_2 exists only when $\alpha_1-\beta_1\delta_1-\beta_1h_2 > \frac{\delta_1+h_2}{1-h_1}$ and $h_1 < 1$.
4. The coexistence equilibrium $E^* = (u^*, v^*, w^*)$, where
$$\begin{aligned}v^* &= \frac{\delta_2+h_3}{\alpha_2-\beta_2\delta_2-\beta_2h_3},\\ w^* &= \frac{(\alpha_1-\beta_1\delta_1-\beta_1h_2)u^*-\delta_1-h_2}{(\alpha_2-\beta_2\delta_2-\beta_2h_3)(1+\beta_1u^*)},\end{aligned}$$
and u^* is the positive root of the following equation
$$\epsilon_1w^2+\epsilon_2w+\epsilon_3 = 0,$$
with
$$\begin{aligned}\epsilon_1 =& \beta_1\beta_2h_3+\beta_1\beta_2\delta_2-\alpha_2\beta_1\\ \epsilon_2 =& \alpha_2\beta_1-\beta_1\beta_2\delta_2-\beta_1\beta_2h_3-\alpha_2-\alpha_2\beta_1h_1\\ &+\beta_2\delta_2+\beta_2h_3+\beta_1\beta_2\delta_2h_1+\beta_1\beta_2\delta_2h_1h_3,\\ \epsilon_3 =& \alpha_2-\beta_2\delta_2-\beta_2h_3-\alpha_1\delta_2-\alpha_1h_3-\alpha_2h_1\\ &+\beta_2\delta_2h_1+\beta_2h_1h_3.\end{aligned}$$

Note that E^* exists only when $\alpha_2 > \beta_2\delta_2+\beta_2h_3$, $(\alpha_1-\beta_1\delta_1-\beta_1h_2)u^* > \delta_1+h_2$, and any one of the $\epsilon_i < 0,\ i = 1,2,3$. To examine the local stability of the equilibrium point, the Jacobian matrix of the system (13) at any point (u,v,w) is evaluated and obtained as follows:

$$J(u,v,w) = \begin{bmatrix} j_{11} & j_{12} & 0\\ j_{21} & j_{22} & j_{23}\\ 0 & j_{32} & j_{33}\end{bmatrix}, \tag{14}$$

where

$$\begin{aligned}j_{11} =& 1-h_1-2u-\frac{\alpha_1v}{(1+\beta_1u)^2},\\ j_{12} =& -\frac{\alpha_1u}{(1+\beta_1u)},\quad j_{21} = \frac{\alpha_1v}{(1+\beta_1u)^2},\\ j_{22} =& \frac{\alpha_1u}{1+\beta_1u}-\frac{\alpha_2w}{(1+\beta_2v)^2}-\delta_1-h_2,\\ j_{23} =& -\frac{\alpha_2v}{1+\beta_2v},\quad j_{32} = \frac{\alpha_2w}{(1+\beta_2v)^2},\\ j_{33} =& \frac{\alpha_2v}{1+\beta_2v}-\delta_2-h_3.\end{aligned}$$

Theorem 4. *(i) The equilibrium E_0 is always unstable.*

(ii) The equilibrium E_1 is stable if $\delta_1+h_2 > \frac{\alpha_1(1-h_1)}{(1+\beta_1(1-h_1))}$.

(iii) If the equilibrium E_2 exists and $(1+\beta_1u_2)^2 > \alpha_1\beta_1v_2$ and $\delta_2+h_3 > \frac{\alpha_2v_2}{1+\beta_2v_2}$, then E_2 is stable.

(iv) The equilibrium E^* exists and is locally asymptotically stable if $\Omega_1 > 0, \Omega_3 > 0$ and $\Omega_1\Omega_2 > \Omega_3$, where Ω_1, Ω_2 and Ω_3 are given in the proof.

Proof. (i) The eigenvalues of Jacobian matrix (14) at E_0 are $\lambda_1 = 1-h_1, \lambda_2 = -\delta_1-h_2$ and $\lambda_3 = -\delta_2-h_3$. Clearly, $1-h_1$ is always positive, and $-\delta_1-h_2$ and $-\delta_2-h_3$ are always negative. So, E_0 is always unstable.

(ii) At the equilibrium E_1, the Jacobian matrix (14) have following equilibrium points $\lambda_1 = h_1-1, \lambda_2 = \frac{\alpha_1(1-h_1)}{1+\beta_1(1-h_1)}-\delta_1-h_2$ and $\lambda_3 = -\delta_2-h_3$. Clearly, h_1-1 and $-\delta_2-h_3$ are always negative, and λ_2 is negative if $\delta_1 + h_2 > \frac{\alpha_1(1-h_1)}{(1+\beta_1(1-h_1))}$, which assures the local stability of E_1.

(iii) The eigenvalues of Jacobian matrix (14) associated with the equilibrium E_2 are $\lambda_1 = \bar{d}_{33}$ and $\lambda_{2,3} = \frac{\bar{d}_{11} \pm \sqrt{\bar{d}_{11}^2 + \bar{d}_{12}\bar{d}_{21}}}{2}$, where $\bar{d}_{mn}$ are obtained from j_{mn} replacing (u, v, w) by $(u_2, v_2, 0)$ in equation (14). So, the equilibrium E_2 is stable if $(1 + \beta_1 u_2)^2 > \alpha_1 \beta_1 v_2$ and $\delta_2 + h_3 > \frac{\alpha_2 v_2}{1+\beta_2 v_2}$.

(iv) The characteristic equation of (14) corresponding to E^* is

$$\lambda^3 + \Omega_1 \lambda^2 + \Omega_2 \lambda + \Omega_3 = 0,$$

where $\Omega_1 = -d_{11} - d_{22}$, $\Omega_2 = d_{11}d_{22} - d_{12}d_{21} - d_{32}d_{23}$, $\Omega_3 = d_{11}d_{23}d_{32}$. Here, d_{mn} are obtained from j_{mn} replacing (u, v, w) by (u^*, v^*, w^*) in equation (14). Make use of Routh-Hurwitz (R-H) criterion, the equilibrium point E^* is locally asymptotically stable $\Omega_1 > 0, \Omega_3 > 0$ and if $\Omega_1 \Omega_2 > \Omega_3$ holds. $\square$

6.2. Numerical Simulations

The model (13) exhibits chaotic dynamics for following set of parameter $\alpha_1 = 5$, $\alpha_2 = 0.1$, $\beta_1 = 3.0$, $\beta_2 = 2, \delta_1 = 0.4, \delta_2 = 0.01$, $h_1 = 0.01, h_2 = 0.02, h_3 = 0.001$, see Fig. 11. To explore the impact of harvesting parameters on the system dynamics, in Figs. 12-14, we plot the bifurcation diagram and fluctuation of the LLE with respect to h_1, h_2, and h_3 and fix all other parameters as previously mentioned. If we increase the value of h_1 from 0.0 to 0.2 and keep all the other parameters the same, the chaotic dynamics of the system change into stable behavior through period-halving oscillation, which is represented in Fig. 12a. When the value of h_1 is large, the system settles into a stable focus, whereas when the value of h_1 is low, the chaotic dynamics in the system remain the same. We have also noted that above a critical value of $h_1 = 3.5$, the top-predator population becomes extinct, while prey and the middle-predator show oscillatory behavior. Now, with the increase of h_2, the system becomes into stable state from its chaotic dynamics via period-halving oscillation, see Fig. 13a. Note that the density of middle-predator decreases as h_2 increases, which leads to density of prey population increase and top-predator decrease due to unavailability of food. This phenomena can control chaos. Similarly, the system becomes stable from its chaotic dynamics through with an increase in harvesting parameter h_3, see Fig. 14a. The chaotic dynamics of the system (13) with respect to the harvesting parameters h_1, h_2, and h_3 are confirmed through the positive LLE values (see Figs. 12b, 13b, and 14b). According to our numerical simulations, by increasing the harvesting parameters and fixing all other parameters, the system switches from chaos to a limit cycle oscillation, then settles into a steady state solution.

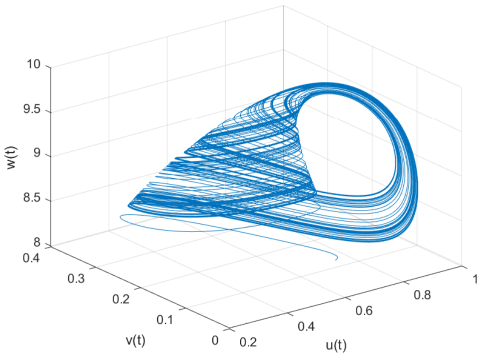

Fig. 11. The phase portrait of the model (13) for the set of parameters given in Fig. 2 with $h_1 = 0.01, h_2 = 0.02$ and $h_3 = 0.001$.

7. Conclusions

Most ecological systems that have been modeled reveal chaotic behavior because they are nonlinear. Chaotic systems always react roughly because of their sensitive dependency on their system parameters and initial conditions. Therefore, it must be controlled in order to predict how the ecological system will develop in the future. Hastings-Powell first found chaos in a model of a food chain with three species. Hastings-Powell first found chaos in a model of a food chain with three species. This model has been widely supported by the scientific community because of its use in describing the dynamics of numerous ecological systems. In this review article, we investigate the controlling mechanisms of chaos in the H-P model through various ecological factors. The ecological factors used are as follows: (i) providing additional food to predators; (ii) alley effect in prey growth; (iii) prey refuge; (iv) the fear effect; and (v) species harvesting. The ecologically possible equilibrium points and corresponding local stability are studied for all cases. In Section 3, we showed that the stability of the H-P model is controlled by the quality and quantity of supplementary food for predators. The interaction between prey and predator decreases as the predator's food supply increases. Also, it has been found that any amount of food has a negative effect on the ecosystem and can lead to the extinction of a population. From Figs. 6 and 7, it is clear that the Allee effect and prey refuge parameters play an important role in regulating the chaotic dynamics of the H-P model. As prey refuges expand, the availability of prey to middle predators decreases, causing all species to maintain positive densities. It should be noted from Section 5 that as fear parameters increase, the growth rate of prey and middle predators is limited, which leads to a lower number of prey and middle predators being consumed by the middle and top predators in the system. This situation allows all three species to coexist and maintain a positive density level. Figures 12-14 show that harvesting parameters have a stabilizing effect on system dynamics. Thus, with the proper choice of harvesting parameters, one can keep the chaos under control in the H-P model. So, we can conclude that the introduced chaos control strategies in the H-P model are non-chemical and easy to implement. The methods derived in this paper are very helpful for protecting biodiversity and managing ecosystems. The ideas discussed in this study can be extended by considering the H-P model with various functional responses, which leaves this as future work. The same ecological characteristics that have been used in this paper can be extended to various eco-epidemiological mod-

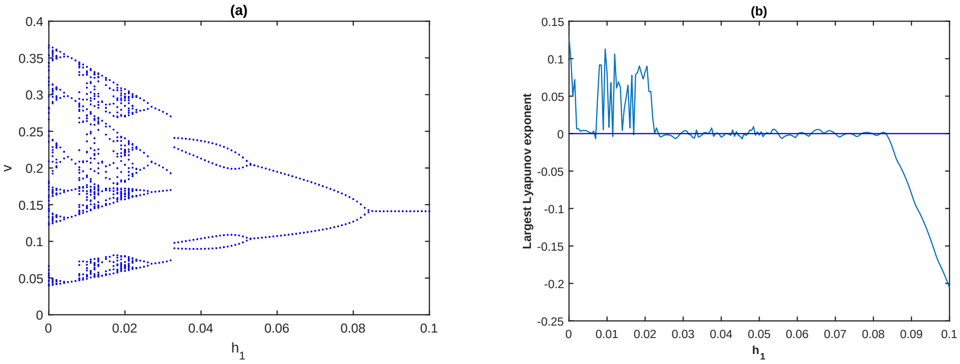

Fig. 12. For system (13): (a) a bifurcation diagram of the middle-predator with respect to parameter h_1; (b) fluctuation of the LLE with respect to h_1.

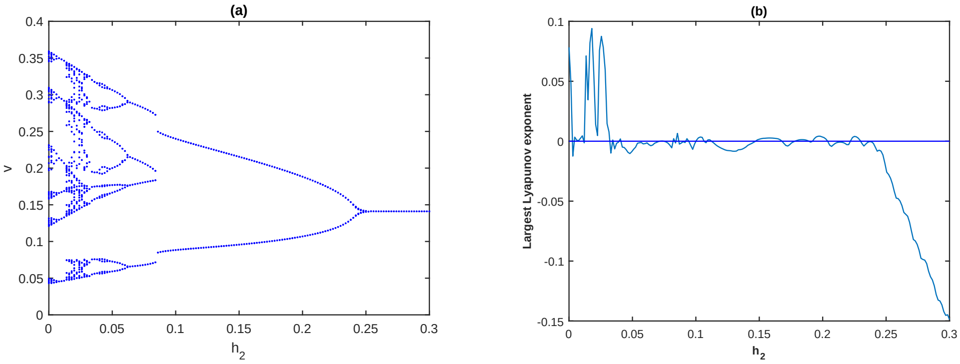

Fig. 13. For system (13): (a) a bifurcation diagram of the middle-predator with respect to parameter h_2; (b) fluctuation of the LLE with respect to h_2.

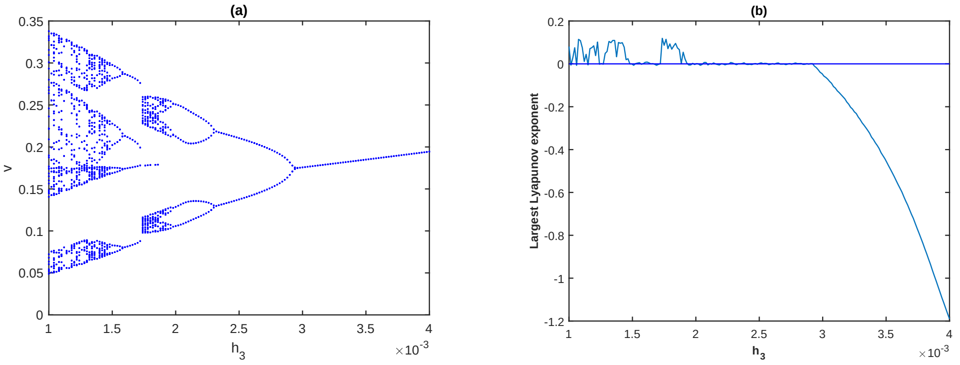

Fig. 14. For system (13): (a) a bifurcation diagram of the middle-predator with respect to parameter h_3; (b) fluctuation of the LLE with respect to h_3.

els and fisheries models, which also leaves room for future work.

Acknowledgments

The first author would like to thank the second-century fund (C2F), Chulalongkorn University for their funding support. This work is supported by the Intelligent Control Automation of Process Systems Research Unit, Ratchadaphiseksomphot Fund, Chulalongkorn University.

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