Engineering Journal issn: 0125-8281

Review

Recent Advances and Applications of Fractional-Order Neural Networks

Monalisa Maiti[1], Sunder M[2], Abishek R[2], Kishore Bingi[2,*], Nagoor Basha Shaik[3], and Watit Benjapolakul[3]

1 School of Computer Science and Engineering, Vellore Institute of Technology, Vellore, India
2 School of Electrical Engineering, Vellore Institute of Technology, Vellore, India
3 Artificial Intelligence, Machine Learning, and Smart Grid Technology Research Unit, Department of Electrical Engineering, Faculty of Engineering, Chulalongkorn University, Bangkok 10330, Thailand
*E-mail: bingi.kishore@ieee.org (Corresponding author)

Abstract. This paper focuses on the growth, development, and future of various forms of fractional-order neural networks. Multiple advances in structure, learning algorithms, and methods have been critically investigated and summarized. This also includes the recent trends in the dynamics of various fractional-order neural networks. The multiple forms of fractional-order neural networks considered in this study are Hopfield, cellular, memristive, complex, and quaternion-valued based networks. Further, the application of fractional-order neural networks in various computational fields such as system identification, control, optimization, and stability have been critically analyzed and discussed.

Keywords: Cellular networks, control systems, fractional calculus, Hopfield networks, identification, memristive neural networks.

ENGINEERING JOURNAL Volume 26 Issue 7
Received 4 February 2022
Accepted 18 July 2022
Published 31 July 2022
Online at https://engj.org/
DOI:10.4186/ej.2022.26.7.49

Nomenclature

BAM	Bidirectional Associative Memory
CMOS	Complementary Metal-Oxide Semiconductor
CNN	Cellular Neural Network
CVNN	Complex Valued Neural Networks
FFNN	Feed Forward Neural Network
FOBAM	Fractional-order BAM Neural Network
FOCNN	Fractional-order Cellular Neural Network
FOCVNN	Fractional-order Complex Valued Neural Networks
FOHNN	Fractional-order Hopfield Neural Networks
FOMNN	Fractional-order Memristive Neural Networks
FONN	Fractional-order Neural Network
FOQVNN	Fractional-order QVNNs
FOQVNNLDD	Fractional-order QVNNs with Leakage and Discrete Delays
FOQMNN	Fractional-order Quaternion-Valued Memristive Neural Networks
FOQVBAM	Fractional-order Quaternion-Valued Bidirectional Associative Memory
LKF	Lyapunov-Krasovskii functional
LMI	Linear Matrix Inequality
MFOBNN	Memristor-based Fractional-order BAM Neural Network
MVM	Matrix-Vector Multiplication
NARX	Nonlinear Auto-regressive Exogenous Network
NN	Neural Network
PSO	Particle Swarm Optimization
QVNN	Quaternion-Valued Neural Networks
ReLU	Rectified Linear Unit

1. Introduction

In recent years it has been proven that combining fractional calculus and neural networks will give higher efficiency in the overall model [1]. Compared to conventional NN, the FONN describes the properties of the neurons such as memory and hereditary of various systems effectively. This effectiveness is because the fractional-order systems have infinite memory [2, 3, 4]. Also, it is to highlight that the computation ability of FONN gives efficient data processing, stimulus anticipation, and phase shift during the oscillatory neurons firing. Therefore, the fractional-order form of neural networks has been an excellent and powerful tool in various computational fields. Recently, the authors of [5] have proposed the fractional-order form of ReLU activation functions and its variants. The developed activation functions have shown better performance compared to conventional. Thus, this study provides a critical review of recent advances in neural network structures such as Hopfield, cellular, memristor-based, complex-valued, etc. Also, a review of applications of these networks for system identification, control, stability, optimization, etc., is presented.

The remaining sections of the paper are organized as follows: Section 2 presents the recent advances in FONNs. The application of FONNs in various fields is described in Section 3. Finally, the conclusions and future directions are given in Section 4.

2. Advances in Fractional-Order Neural Networks

This section presents the critical review on various advances in the fractional-order form of Hopfield, Cellular, Memristive, Complex-Valued, and other neural networks.

2.1. Feed Forward Neural Networks

FFNN is one of the most straightforward variants of neural networks as they pass information in one direction through various input nodes until it makes it to the output node [6, 7, 8]. On the other hand, RNN is more complex than FFNN. Here, each node in the RNN acts as a memory cell, continuing the computation and implementation of operations [9]. The networks save the output of processing nodes and feed them back into the model, and hence they did not pass the information in one direction only [10].

In 2011, the researchers developed an FFNN with PSO to solve differential equations [11]. Later, the work in [12] established FFNN based on sine and cosine, which helped solve fractional wave and boundary value problems. In 2020, the researchers considered fractional derivatives based on Caputo's derivative to develop a method for optimizing fractional-order delay optimum control problems [13]. Also, algorithms such as using Grünwald-Letnikov's derivative function have been equipped to model an FFNN that focuses on reducing the number of parameters required for system identification. To study the dynamical behavior of a multi-stable fractional-order system, in 2021, the method of fractional-order behavior of robotic manipulators has been equipped with the wolf algorithm [14]. Also, another reported work in [15] has used Grünwald-Letnikov's derivative function as a learning algorithm and fractional calculus to reduce convergence error and convergence speed. The single-point search algorithm has made an efficient global learning machine to determine the optimal search path [16].

2.2. Fractional-Order Neural Networks

In [17], the work deals with the discrete FONN with a Fuzzy Lyapunov synthesis approach has been reported. Here, a non-linear discrete FONN algorithm has been implemented to apply this proposed NN in modeling a heat transfer. In [18], the paper investigates a class of FONNs. Initially, α-exponential stability is introduced along with a novel differential inequality-based fractional-order method for analyzing the network's stability. The work also proposes the use of fractional ordering for chaotic network synchronization. The equilibrium point's existence and α-exponential stability have been considered based on investigations. The work in [19] discusses a class of FONNs with delay. This work establishes an adequate condition for the uniform stability of such FONNs with a delay. Also, the equilibrium point's uniqueness, existence, and stability have been shown.

Further, in [20], the paper investigates a category of FONNs for their finite-time stabilities. It is worth highlighting that sufficient conditions have been introduced to establish the time-stability of such a class of FONNs with the Caputo fractional derivatives. These conditions are based on the Gronwall theorem, Laplace transforms, and Mittag-Leffler approximation approaches. Here, the asymptotic stability results of such FONNs are also shown. In [21], a category of FONNs with delayed systems has been scrutinized for their finite-time stabilities. It has been observed that the inequality technique is used to obtain two novel sufficient conditions that are delay-dependent, which ensures the stability of such FONNs. Also, the Caputo derivative has been derived and verified with Cauchy-Schwartz inequality. In [22], the paper focuses on the FONNs based on Riemann-Liouville that have discrete and distributed delays and analyses their asymptotic stability. Here, two sufficient conditions are derived by constructing a suitable Lyapunov functional to ensure that the addressed NN is stable asymptotically.

In [23], stability analysis of FONNs with and without delay using fractional approaches such as Lyapunov and Razumikhin has been reported. It has been observed that the non-linear constraints were handled using the S-procedure to get a more comprehensive selection of the systems. The uniqueness and existence of the point of equilibrium have been proved in this work. Unlike the last, in [24], the article focuses on FONN with double leakage delays and analyzes their stability and bifurcation. This is done by considering leftover delay as a bifurcation parameter after intercalating one delay. It has been observed that incongruent critical values have been gained for different delays-induced bifurcations. Also, the implementation of Hopf bifurcation can be seen in this paper. In [25], the problem of parameters synchronization and identification of FONNs with time delays have been investigated. It has been observed that some analytical techniques and an adaptive control method were designed here to synchronize the two uncertain, complex networks with time delays.

Further, [26] investigates the global projective synchronization of FONNs by combining open-loop control and adaptive control. It is worth highlighting that based on Caputo's fractional derivation, a novel fractional-order differential inequality has been derived in this paper to establish the monotonicity of the continuous and differential functions. It also applies several control strategies to ensure complete synchronization, anti-synchronization, and stabilization of the addressed NNs. Contrary to [26], in paper [27], the focus is on the non-identical FONN based on the sliding mode control technique. Here, the global projective synchronization of such FONNs is investigated. It has been observed that the authors gave derived synchronization conditions with uncertainty parameters for memristor-based NNs with fractional-order multiple time-delays that are based on Filippov's solution and inclusion theory. In [28], it has been observed that adaptive projective synchronization of time-delayed FONNs has been analyzed. Here, efficient hybrid control strategies are designed for delayed FONNs with uncertain parameters using active and adaptive control methods. Adaptive synchronization-based method and Adams-Bashforth-Moulton predictor-corrector scheme have been implemented.

The paper in [29] studies the synchronization for a category of uncertain FONNs that have disturbed parameters. In this paper, an adaptive synchronization controller is designed based on the fractional-order extension of the Lyapunov stability criterion. Also, a fractional-order adaptation law is proposed to update the controller parameter online. The work in [30] is based on Global Mittag-Leffler's synchronization in FONNs with discontinuous activation functions. Here, singular Gronwall inequality with properties of fractional calculus is used to prove the existence of a global solution under the framework of Filippov for FONN with discontinuous activation functions. Also, the framework of the Filippov solution for such NNs following Caputo's fractional derivative is given. This is based on the

DOI:10.4186/ej.2022.26.7.49

Lyapunov stability criterion. This paper also provided a clue to study FONNs with discontinuous functions. Also, some requirements have been established to ensure the asymptotic stability of FONNs based on the Lyapunov function's fractional derivative. In [31], a class of NNs with fractional-order derivatives is investigated. Here, some new conditions based on Krasnoselskii's fixed point theorem and inequality technique are employed to ensure the uniqueness and existence of the non-trivial solution. Also, the stability of the FONNs in fixed time intervals is proposed here. In [32], LMI stability conditions for linear and non-linear fractional-order systems have been presented. The global stability analysis of FONNs is employed, and the results are obtained from LMI conditions. The two-norm and fractional-order Lyapunov direct method has been implemented in this paper. Now, in [33], the article focuses on the parameter estimation problem of FONNs. Here, the identification based on the synchronization method has been generalized by applying parameter update laws and adaptive control. Also, the technique has been used for parameter identification and synchronization simultaneously.

2.3. Fractional-Order Hopfield Neural Networks

The architecture of the Hopfield network model is shown in Fig. 1. The model contains several linear and non-linear elements, with the size of the network being N. The block 'A' in the circuit simulates biologic neurons' synapse and non-linear characteristics. At the same time, block 'B' is used to simulate the biologic neurons' time-delay characteristics. The governing equations of the model are given as [34],

$$C_i \frac{\mathrm{d}}{\mathrm{d}t} P_i = \sum_{j=1}^{N} \left(\frac{1}{R_{ij}}\right) V_j - P_i \left(\frac{1}{R_{i0}} + \sum_{j=1}^{N} \frac{1}{R_{ij}}\right) + I_i, \tag{1}$$

$$P_i = \left(\frac{1}{\lambda}\right) \varphi_i^{-1} V_i, \tag{2}$$

where, P_i denotes the input of operational amplifier at i^{th} neuron, V_i represents the output of operational amplifier at i^{th} neuron, and λ is the learning rate.

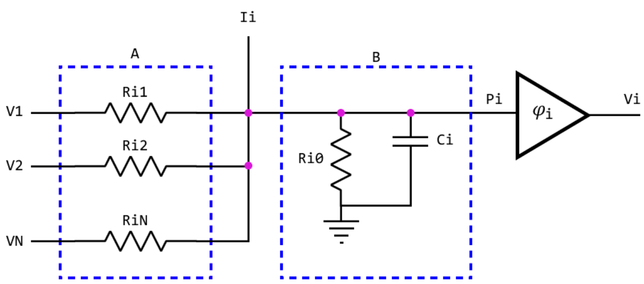

Fig. 1. Architecture of the Hopfield network model.

The governing equations for the fractional-order form of Hopfield network model is obtained from Eqs. (1) and (2) as follows:

$$C_i \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} P_i = \sum_{j=1}^{N} \left(\frac{1}{R_{ij}}\right) V_j - P_i \left(\frac{1}{R_{i0}} + \sum_{j=1}^{N} \frac{1}{R_{ij}}\right) + I_i, \tag{3}$$

$$P_i = \left(\frac{1}{\lambda}\right) \varphi_i^{-1} V_i, \tag{4}$$

where $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}$ is the fractional-order derivative of order $\alpha \in (0, 1)$.

The first attempt to undertake Hopf bifurcation phenomena in FONN by theoretically characterizing has been developed in 2011 [35]. The authors have replaced fractance in the typical capacitor from the Hopfield NN, which formed a familiar FOHNN model. In 2012, they put forward a theoretical stability analysis by characteristic parameters for the FONN of Hopfield type [36]. The work observed that the FONN might show rich dynamical behavior. Their dynamics become increasingly complex as the fractional-order rises, leading to the chaotic behavior of the system. In [37], the Mittag-Leffler function has been proposed to synchronize a Class of Fractional-Order Chaotic NN. The theoretical results obtained are applied to a FOHNN model. During the same year, the complex dynamical behaviors of such networks have been inspected and observed that an expansive range of interesting dynamical phenomena has been found to exist. In 2014, to present a theoretical stability analysis of FOHNN with time delays, two FOHNN with different ring structures and delays were introduced; the derivation for the adequate conditions for stability of the systems are also provided [38]. In 2015, Shuo Zhang et al. investigated their solutions' existence and stability conditions within the frame of Filippov solutions. Under the proposed requirements, the uniqueness of the equilibrium point of the system is broken down [39]. In 2016, the global Mittag-Leffler stability condition presented linear matrix inequalities by applying the fractional Lyapunov method with impulses [40].

2.4. Fractional-Order Cellular Neural Network

In CNN, the cell is the fundamental circuit element, as shown in Fig. 2. It is to note that any CNN cell is connected only to its neighbor cells. For the circuit, the input, state, and output equations are given as follows [41]:

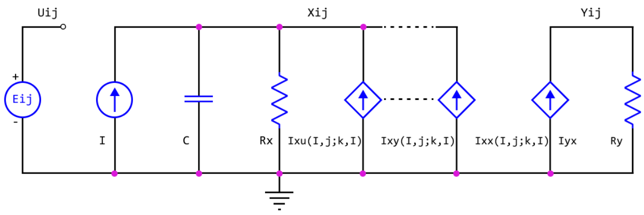

Fig. 2. Architecture of the cellular network model.

$$u_{ij}(t) = E_{ij}, \tag{5}$$

$$\begin{aligned} C\frac{\mathrm{d}}{\mathrm{d}t}x_{ij}(t) = &-\frac{1}{R_x}x_{ij}(t) \\ &+ \sum_{C(k,l)\in N_r(i,j)} A(i,j;k,l)y_{kl}(t) \\ &+ \sum_{C(k,l)\in N_r(i,j)} B(i,j;k,l)u_{kl}(t) \\ &+ C(i,j;k,l)x_{kl}(t) + I, \end{aligned} \tag{6}$$

$$y_{ij}(t) = \frac{1}{2}(|x_{ij}(t)+1| - |x_{ij}(t)-1|), \tag{7}$$

where $C(i,j)$ is the cell of i^{th} row and j^{th} column, u_{ij}, x_{ij} and y_{ij} are the input, state and output voltages respectively of each neighbor cell $C(k,l)$. The voltage controlled current sources are given as,

$$I_{xy}(i,j;k,l) = A(i,j;k,l)y_{kl}, \tag{8}$$

$$I_{xu}(i,j;k,l) = B(i,j;k,l)u_{kl}, \tag{9}$$

$$I_{xx}(i,j;k,l) = C(i,j;k,l)x_{kl}. \tag{10}$$

In (8), (9) and (10), A, B and C are coefficients known as cloning templates. In (6) is given as,

$$N_r(i,j) = C(k,l) : \max(|k-i| - |l-j|) \le r. \tag{11}$$

From the above circuit in Fig. 2, the governing equations for the fractional-order form of CNN are given as follows:

$$u_{ij}(t) = E_{ij}, \tag{12}$$

$$\begin{aligned} C\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}x_{ij}(t) = &-\frac{1}{R_x}x_{ij}(t) \\ &+ \sum_{C(k,l)\in N_r(i,j)} A(i,j;k,l)y_{kl}(t) \\ &+ \sum_{C(k,l)\in N_r(i,j)} B(i,j;k,l)u_{kl}(t) \\ &+ C(i,j;k,l)x_{kl}(t) + I, \end{aligned} \tag{13}$$

$$y_{ij}(t) = \frac{1}{2}(|x_{ij}(t)+1| - |x_{ij}(t)-1|), \tag{14}$$

where $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}$ is the fractional-order derivative of order $\alpha \in (0,1)$.

There have been various advancements in FOCNN in techniques and learning algorithms that have potential applications. In 2012, to establish secure communications by improving the security of the chaotic communication system, FOCNN had been proposed with Fractional-order four-cell CNN [42]. Then in 2014, fractional Lyapunov method and Mittag-Leffler functions provided leverage to the development of FOCNN with time-varying delays [43]. Although in 2018, to deal with multiple-time delays and fractional-order linear delayed systems, fractional-order adaptive laws have been used to examine the stability theory in fractional linear-delayed systems for controlling FONNs with or without sector nonlinearities [44]. As advancement in this domain, fractional-order differential inequality, including time delays, has been proposed in 2020 with a feedback controller to inspect the multi-weighted complex structure, robust synchronization of on FOCNN under linear coupling delays [45]. Researchers in 2021 have used the contraction mapping principle with FOCNN to analyze S-asymptotically w-periodic oscillations in FOCNN [46]. A fractional-order system of a multidimensional-valued neural network is split into four or two fractional-order systems of real-valued neural networks based on Hamilton rules. A novel inequality comprising a quadratic term is inferred and analyzed on the synchronization and stability problem [47]. Also, LKFs are constructed from these new inferred inequalities with quadratic coefficients.

2.5. Fractional-Order Memristor-Based Neural Networks

A memristor is a hardware and electrical component which performs the same function as a resistor but remembers the amount of charge that has previously flowed through it. This hardware's specific vital properties are exploited using complex and computational intensive neuromorphic systems such as CNNs and their derivatives. Memristors are two-terminal nano solid-state switching devices. They tend to fix the issues of memory wall and communication bottlenecks raised from the conventional CMOS technology. With the Non-Volatility property of this memristor, the capability to retain memory even without power, a memristor crossbar array architecture is proposed for practical use in real-world applications because of its high density and parallel computational ability. Memristors allow in-memory computing. They accelerate MVM, search, and bitwise operations. Most NN architectural layers include a convolutional layer whose computation pattern is MVM, and the fully-connected layer is memory-intensive. Using a memristor allows efficient hardware acceleration of CNN and other variants of NNs [48, 49, 50, 51].

In 2014, the authors proposed network stability conditions by applying an inequality approach and analysis method [52]. Further, proposed adequate requirements to ensure the equilibrium point is existing, unique, and stable. In the same year, Jiejie Chen et al. introduced memristor-based FONN, the dynamic behavior of a class of FOMNN has been proposed [53]. Moreover, sufficient criteria for the uniqueness of the equilibrium point synchronization of the networks are also proved. In the same year, using approaches like Gronwall inequality, Laplace transforms, and estimates of Mittag-Leffler functions, the finite-time stability criterion of Fractional-Order distributed delayed BAM NN have been proposed [54]. In [55], within the frame of the Filippov solution and using differential inclusion theory, a comparison theorem has been presented for a set of fractional-order systems with multiple time-delays.

The presented requirements for attractivity for memristor-based FONN and global boundedness with non-Lipschitz activations. Further, the derivation for synchronization conditions of the system as mentioned above with parameter uncertainty is shown. Additionally, a development condition to make sure the solutions exist has been introduced by Shuo Zhang et al. [56]. In 2017, Xujun Yanget et al. proposed Quasi-uniform synchronization of delayed FOMNN and several adequate criteria to ensure the quasi-uniform synchronization for the FOMNN with delay is put forward [57]. In 2018, Master-Slave synchronization for some MFOBNN with mixed time-varying delays and switching jumps mismatch had been proposed. Some new projective lag synchronization criteria have been conferred for such networks obtained by differential inclusions theory, set-valued map, fractional Barbalat's lemma, and control scheme [58]. In 2019, proposed a generic fractional Halanay inequality with multiple time-varying delays [51]. The delayed FOMNN is discussed by capitalizing this new upper bound and presenting inequality based on the Lyapunov function method. In the same year, Hong-Li Li et al. handled FOQVNNLDD without any disintegration [59]. The work transformed the FOQVNNs into real-valued systems, which is more concise and naturalistic than the actual decomposition method.

Two novel methods are proposed in [60] for the synchronization problem of FOQVNN. The first includes a new establishment of LKFs and fractional-order derivative inequality. The latter method equips both norm comparison rules and generalized Gronwall-Bellman inequality with the help of the Laplace transform of the Mittag-Leffler function. A Lyapunov stability theory and Caputo fractional derivative with several algebraic criteria are established in [61] for guaranteeing the finite-time Mittag-Leffler synchronization of FOQVNNs. This method has shown less conservatism than existing results. To decrease the computational complexity and avoid the non-commutativity of quaternion multiplication, FOQVNN is separated into four real-valued FONNs. Inequality techniques and Lyapunov functional criteria are proposed in [62] to ensure Mittag-Leffler stability and impulsive control. The principle of homeomorphism, the Lyapunov fractional-order method, and the matrix inequality approach are applied in [63] to derive sufficient conditions to study the global asymptotic stability problem for the FOQVBAM neural network. The technique has shown better results confirming the existence, uniqueness, and global asymptotic stability of the system's equilibrium point.

The global synchronization criteria are first derived by employing the second method of Lyapunov. A new set of FOQMNNs and a unique scientific expression for the quaternion-value inductance has been proposed following its features. A new sufficient criterion has been presented to verify the finite-time stability of the FOQMNN system by ignoring the impulsive effects by employing the Laplace transform in the same year. In 2020, proposed a synchronization for commensurate Riemann-Liouville FOMNN with unknown parameters through Lyapunov's second method. Further, two inequalities of Riemann-Liouville fractional derivatives that play a vital role are provided to theoretically analyze the Riemann-Liouville fractional differential system [64]. Similarly, a fractional–based theorem has been introduced to deal with the effects of reaction-diffusion and time delay of FOMNN [49].

Additionally, the activation function has been extended to sporadic cases on the Filippov solution and set value mapping theory. In addition, a finite-time projective synchronization of FOMNN with mixed time-varying delays and several criteria to ensure that along with the set-valued map and frame of fractional-order differential inclusion has been proposed by Meng Hui et al. [48]. Also, three properties are established, which significantly increases some results of settling time of FOMNN.

In [65], a new model is designed to counter the fixed-time synchronization problem. The approach takes two general inequalities, extended Cauchy-Schwarz inequality and generalized derivative of the fractional-order absolute value function, to realize the analysis with the acquisition of this less conservative fixed time. Further, two new inequalities to deal with global Mittag-Leffler synchronization are deduced and compared with the existing inequalities in [66]. Also, with Lyapunov theory, two more novel functionals are constructed along with multiple more flexible criteria and a numerical example to demonstrate the proposal's effect on this problem. Similarly, a benchmark technique is used in [67] for the delayed fractional-order systems of multidimensional-valued memristive neural networks stability. The proposed Lyapunov method with several new lemmas has shown many advantages, including lower conservatism and higher flexibility. Using impulsive and effective analysis techniques, the system's dynamic behavior is studied in [68]. The Lyapunov method obtains some specific conditions to ensure the stability and passivity of the Memristor-based fractional-order competitive neural networks. To counter stability issues, the technique in [69] uses the framework of Filippov solutions with suitable Lyapunov-functional and differential inclusion theoretical analysis. By employing specific order of value, fractional-order stability of orders 0 to 1 and 1 to

2 are discussed separately by creating a sufficient criterion using Laplace transform and Mittag-Leffler function and Generalized Gronwall inequality. Original fractional-order quaternion-valued systems are divided into four fractional-order real-valued systems based on the non-commutativity of quaternion multiplication. Sufficient conditions are established in [70] by employing Lyapunov fractional-order derivative, fractional-order differential inclusions, global Mittag-Leffler stability, and set-valued maps.

2.6. Fractional-Order Complex Valued Neural Networks

Stability analysis of FOCVNN has been studied over the years. In 2014, a FOCVNN had been proposed using fractional-order differential equations to examine the stability of FOCVNN using memristor-based NNs [71]. In the same year, the work in [72] has presented a complex-valued recurrent NN to find a proper activation function in various complex processes. Continuous FOCVNN with a time delay algorithm has been equipped in the model. In [73], the dissipativity of NNs with time delay is analyzed using a FOCVNN model developed with Lyapunov functions and fractional Halanay inequality and Caputo fractional-order differential equations, which helps in the study of linear, non-linear systems and economic systems.

2.7. Other Neural Networks

2.7.1. Fuzzy-based Neural Networks

In [74], proposed a T-S fuzzy-based NN model using fuzzy Lyapunov synthesis. This paper equipped the Adams-Bashforth-Moulton algorithm and synchronized fractional-order Duffing-Holmes chaotic system. Also, as an advancement in this domain, Recurrent Fuzzy NN is proposed in 2020, which uses sliding mode control and fractional calculus [75]. This model helps in fractional sliding mode control for the micro gyroscope. A fractional-order learning algorithm using Grünwald-Letnikov's and Riemann-Liouville's definition is used to develop the model where a two-loop recurrent fuzzy NN is employed to approximate the system uncertainties.

2.7.2. Feedback

In [76], the paper has proposed fractional-order modeling, along with an analysis of stability and control of two robotic manipulators. The pole placement method has been used to derive the control law. Here, the research in the time domain is done using the Mittag-Leffler approach.

2.7.3. Backpropagation

In [77], the screen variables have been screened using the adaptive lasso method, and the various neural network models are established by choosing variables for the seven countries. This paper aims to screen the economic growth in these seven countries, and the local quadratic approximation algorithm has also been implemented.

2.7.4. Forward-Rolling Empirical Decomposition Method

The Forward-Rolling Empirical Decomposition method algorithm uses the fractional calculus predicting model and principal component analysis method to propose a stochastically new financial stock market analysis and financial pricing of stocks model based on Taylor's formula [78].

2.7.5. NARX Neural Network

In [79], proposed a NARX Neural Network model using fractional-order calculus and a backpropagation algorithm. This model helps in the reduction of large-scale systems with fractional-order non-linear structures. Also, a new type of update rule using fractional calculus is obtained, which has provided good results in fractional-order system model reduction [80].

2.7.6. ResNet

In 2019, introduced a generalization methodology for automatically selecting the activation functions inside NN, which led the neurons within the network to adjust their activation functions to precisely fit the input data and considerably reduce the output error [81].

2.7.7. Multilayer Perceptron Network

By using CNN and PSO, in 2019, Brazilian researchers equipped the fractional-order Darwinian PSO segmentation algorithm K-fold cross-validation technique to propose a method recognizing plotting polynomials using a hybrid segmentation method by fractional calculation [82].

2.7.8. Inverse and Forward Problem

In [83], a variable-order based fractional model is proposed. This model can predict the mean fluid velocity profile, and Reynolds stresses of the fluid with accuracy better than 1% for any range of Reynolds number. Direct numerical simulation data and a Physics-Informed Neural Network have been used to obtain the fractional-order.

DOI:10.4186/ej.2022.26.7.49

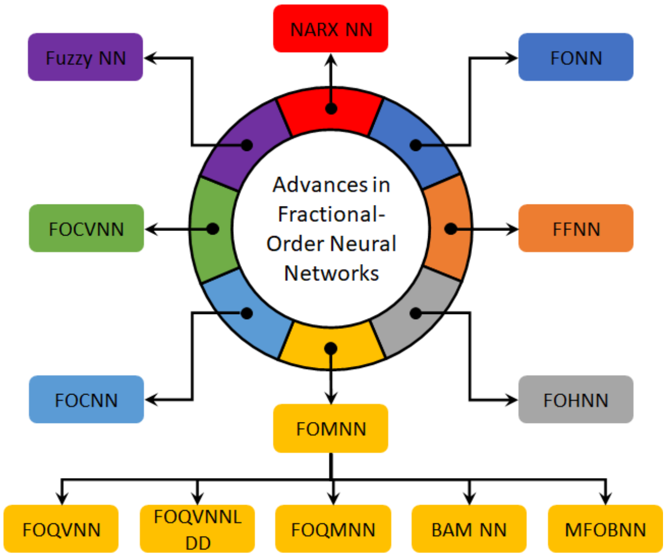

Fig. 3. Classification of various fractional-order neural network structures.

The summary of these advances in fractional-order neural networks in terms of structure, learning algorithm, and others is given in Table 1. From the table, the classification of various fractional-order neural network structures is shown in Fig. 3. Further, the different types of stability analysis approaches used in the literature are summarized in Fig. 4.

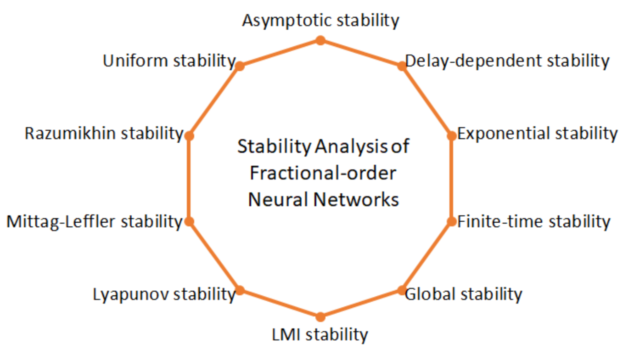

Fig. 4. Various stability analysis approaches used for fractional-order neural networks.

3. Applications of Fractional-Order Neural Networks

This section presents the critical review on the application of FONNs in various fields such as system identification, control, optimization, stability, and synchronization.

3.1. System Identification and Control

In 2011, a T-S fuzzy-based NN was proposed to synchronize fractional-order Duffing-Holmes chaotic systems [74]. The researchers in 2012 used α-exponential stability. They produced a novel fraction order differential inequality that helps in information processing, stimulus anticipation, and frequency independent shifts of oscillatory neural firing [18]. A FOHNN based controller synchronizes a class of Fractional-order Chaotic Neural Networks [37]. In [25], a controller investigating the parameter identification and synchronization problem of FONN with time-delays has a direct behavior of the non-linear dynamical systems. In 2014, the authors studied an Impulsive NN of fractional-order Caputo system of varying time delays and proposed a linear non-impulsive controller [43]. With the help of BAM NN, the Caputo fractional-order BAM neural network is established and analyzed the finite-time stability for pattern recognition and automatic control [54]. In 2016, the neural net's dissipative analysis was studied, which is used to analyze linear, non-linear, and economic systems [73]. In 2017, parameter estimation of NN models was proposed using adaptive control methods [33]. FONN with multiple time-delays is investigated, and some stability criteria are derived with which it is used to control FONNs with or without sector nonlinearities in control inputs [44]. In 2019, fractional calculation and classification proposed a hybrid segmentation method through CNN to recognize simple handwritten polynomials [78]. In [88], the paper researches the stability and bifurcation of a FONN with double delays in application to system control. Also, in the same year, the authors of [83] have used direct numerical simulations data and designed a physics-informed neural network to obtain an order that helps in a turbulent Couette flow system. In 2020, the indirect method for solving a class of DFOCPs had been based on the NN approach and optimal control problems [13]. Model reduction of large-scale systems with Fractional-order non-linear structure is discussed, and a new type of fractional update rule is obtained [79]. A simple and more accurate FONN model for system identification with minimal parameters is received in the paper [75].

3.2. Stability Analysis & Synchronization

In [1], the paper analyzes FONNs with double delays for their stability and bifurcation. The bifurcation results of the FONNs have been deduced once the leakage delays are not identical to the communication delays. Further research has been conducted on a FONN with two unequal delays for its stability and bifurcation [24]. It has been obtained by intercalating one delay and taking the remnant delay as a bifurcation parameter. This paper aimed to analyze the NNs with different delays for their bifurcation problem and generalize the derived results to more high-order FONNs with numerous delays. The authors of [23] examine the stability of both FONNs with and without delay using S-procedure.

The analysis of quaternion-valued FONNs with leakage and discrete delays for their synchronization is reported in [59]. The investigation has been carried out by applying the

Table 1. Summary of advances in fractional-order neural networks.

Year	Country	Journal/Conference	Network Structure	Technique/Method	Learning Algorithm	Ref.
2021	China	Neural Networks	FFNN	Fractional and integer gradient	Single-point search algorithm	[16]
2021	India	Mathematical Methods in the Applied Sciences	–	Haar wavelets	Haar wave basis and Caputo fractional derivative	[84]
2021	China	Mathematical Methods in the Applied Sciences	FFNN	Fractional calculus	Fractional-order learning algorithm using Grünwald-Letnikov's derivative function	[15]
2021	China	IEEE Transactions on Fuzzy Systems	Recurrent fuzzy NN	Fractional calculus and sliding mode control	Fractional-order learning algorithm using fractional derivative definitions	[75]
2021	India	Arabian Journal for Science and Engineering	Feedback	Pole Placement	–	[76]
2021	China	Chaos, Solitons & Fractals	FFNN	Fractional behavior of robotic manipulators	Radial basis functions, Wolf algorithm	[14]
2020	China	Soft Computing	NARX neural network	Fractional calculus	Backpropagation algorithm	[79]
2020	China	Complexity	FOMNN	Flexible Lyapunov function	Fractional-order NN	[48]
2020	China	Neurocomputing	CNN	Contraction mapping principle	Fractional-order CNN	[46]
2020	China	Complexity	FOMNN	Fractional partial and comparison theorem	Adaptive algorithm	[49]
2020	China	Neural Processing Letters	FOCNN	Feedback controller	Fractional order differential inequality with time delays	[45]
2020	China	Journal of the Franklin Institute	FOMNN	Lyapunov direct method	Riemann-Liouville FOMNN with unknown parameters	[64]
2020	Mexico	Chaos, Solitons & Fractals	FFNN	Fractional calculus	Fractional-order learning algorithm using Grünwald-Letnikov's derivative function	[85]
2020	China	Chaos, Solitons & Fractals	–	Space-time auto regression and fractional calculus grey model	Auto regression model and fractional calculus grey model	[86]
2020	China	Neural Processing Letters	FONN	Hopf bifurcations	Fractional-order neural networks and bifurcation parameter	[24]
2020	China	Neural Networks	–	FONN	Caputo fractional-order derivative	[1]
2020	China	Mathematics	Back propagation	Adaptive lasso method	Lasso, local quadratic approximation algorithms	[77]
2020	UK	IEEE Conference	CNN and Recurrent NN	Fractional calculus	–	[87]
2020	Iran	Connection Science	FFNN	Fractional dynamical system using a Padé approximation	Fractional derivatives considered in the Caputo sense	[13]

Table 1. Summary of advances in fractional-order neural networks (continued).

Year	Country	Journal/Conference	Network Structure	Technique/Method	Learning Algorithm	Ref.
2020	China	Neurocomputing	FFNN	NN based on sine and cosine	–	[12]
2019	Mexico	IEEE Conference	ResNet	Fractional-order of the derivative	Fractional derivative	[81]
2019	Brazil	IEEE Conference	Multilayer perceptron network	CNN and Fractional-order darwinian PSO	Segmentation algorithm, k-fold cross validation technique	[82]
2019	China	Chaos, Solitons & Fractals	Forward-rolling empirical decomposition method	Fractional calculus prediction model	Empirical decomposition algorithm and principal component analysis method	[78]
2019	Sweden	Fractional Calculus and Applied Analysis	Inverse and Forward problem	Reynolds-averaged Navier-Stokes equation	Stochastic gradient descent algorithm	[83]
2019	China	Journal of the Franklin Institute	–	FONN	–	[88]
2019	China	Neural Processing Letters	FOQMNN	Quaternion-value memductance	A class of FOQMNN with impulsive effect	[50]
2019	China	Neurocomputing	FOQVNNLDD	Lyapunov direct method	FOQVNNLDD	[59]
2019	China	Journal of the Franklin Institute	FOMNN	Interval Matrix Method	Fractional-order differential equations with multiple time-varying delays	[51]
2018	China	Neurocomputing	FOCNN	Stability theory of fractional linear delayed systems	Asymptotic stability of the controlled system. Fractional-order adaptive laws	[44]
2018	China	Neural Networks	FONN	S-procedure	Fractional Lyapunov direct method and fractional Razumikhin-type theorem	[23]
2018	India	Asian Journal of Control	Memristor-based fractional-order BAM NN	Adaptive switching controller	Class of memristor-based fractional-order BAM neural network	[58]
2017	China	Transactions of the Institute of Measurement and Control	FONN	Hybrid control strategies for FONN	Adaptive projective synchronization of time-delayed FONN	[28]
2017	China	Neurocomputing	FOMNN	Quasi-synchronization for FONN	Fractional differential equation and Caputo's derivative	[57]

Table 1. Summary of advances in fractional-order neural networks (continued).

Year	Country	Journal/Conference	Network Structure	Technique/Method	Learning Algorithm	Ref.
2017	China	Journal of the Franklin Institute	FOMNN	Filippov solution and differential inclusion theory	Adams-Bashforth-Moulton predictor corrector	[89]
2017	China	Physica A: Statistical Mechanics and its Applications	FONN	Combination of adaptive control and parameter update law	Adaptive synchronization, Adams-Bashforth-Moulton predictor-corrector	[33]
2017	China	IEEE Transactions on Neural Networks and Learning Systems	FONN	Simplified LMI stability	Two-norm and fractional-order Lyapunov direct method	[32]
2017	China	International Journal of System Science	FONN	Lyapunov function	Riemann-Liouville's fractional derivative, Asymptotical stability of system	[22]
2016	India	Neural Networks	FOCVNN	Fractional Halanay inequality	Caputo fractional derivative	[73]
2016	China	Neurocomputing	FOMNN	Filippov solutions, Lyapunov functions	Mittag-Leffler stability condition	[56]
2016	China	Neurocomputing	FOHNN	Fractional Lyapunov method	Caputo's fractional derivative	[40]
2016	China	Journal of the Franklin Institute	FOMNN	Filippov solution and differential theory	Caputo's derivative, Adams-Bashforth-Moulton predictor-corrector scheme	[55]
2016	China	Neural Networks	FONN	Lyapunov direct methods	Caputo's derivative, Asymptotic stability using Lyapunov's method	[27]
2015	China	Entropy	FONN	Lyapunov stability criterion	Caputo's fractional derivative	[29]
2015	China	Neurocomputing	FONN	Lyapunov function's fractional derivative	Lyapunov's fractional derivative	[90]
2015	China	Neural Networks	FONN	Lyapunov stability criterion	Singular Gronwall inequality and fractional calculus	[30]
2015	China	Neurocomputing	FOHNN	Stability theory of fractional linear delayed systems	Caputo's derivative	[39]
2015	China	Neural Computing and Applications	FONN	Inequality technique	Cauchy-Schwartz inequality	[21]
2014	China	Entropy	FONN	Analytical technique, adaptive control method	FONN with time delay	[25]
2014	Saudi Arabia	Neurocomputing	FONN	Contraction mapping principle	General class of FONN	[31]
2014	India	IEEE Transactions on Neural Networks and Learning Systems	FOCVNN	Complex valued recurrent NN	Continuous FCVNN with time delays	[72]

Table 1. Summary of advances in fractional-order neural networks (continued).

Year	Country	Journal/Conference	Network Structure	Technique/Method	Learning Algorithm	Ref.
2014	India	Nonlinear dynamics	FOCVNN	Fractional-order differential equations	Fractional-order complex-valued memristor-based NN	[71]
2014	China	Abstract and Applied Analysis	BAM NN	Mittag-Leffler function	Fractional-order BAM neural networks with distributed delay	[54]
2014	USA	Nonlinear Dynamics	FOCNN	Fractional Lyapunov method and Mittag-Leffler functions	Impulsive fractional-order CNNs with time-varying delays	[43]
2014	China	Neural Networks	FOMNN	Lyapunov method	FOMNN	[53]
2014	China	Neural Networks	FONN	Combining open loop control and adaptive control	Class of FONN	[26]
2014	China	Neural Networks	FOHNN	Sufficient conditions for stability	–	[38]
2014	China	Abstract and Applied Analysis	Fractional-order BAM NN	A delay-dependent stability condition	Fractional-order BAM neural networks with delays	[52]
2013	China	Neurocomputing	FONN	Stability theorem with time delay	Continuous fractional-order delayed neural network	[19]
2013	China	Communications in Theoretical Physics	FONN	Laplace transform	Fractional-order CNN	[20]
2013	China	Advances in Mathematical Physics	FOHNN	Riemann–Liouville derivative	FOHNNs with time delay	[91]
2013	China	Entropy	FOHNN	Mittag-Leffler function	Continuous fractional-order cellular neural networks	[37]
2012	China	Neurocomputing	FOCNN	Fractional-order four-cell CNN	Fractional-order four-cell CNN	[42]
2012	USA	Neural Networks	FOHNN	Characteristic parameters	Fractional-order neural networks of Hopfield type	[36]
2012	China	Neural Networks	FONN	α-exponential stability and synchronization	Drive-response fractional-order chaotic neural networks	[18]
2011	Taiwan	Chaos, Solitons & Fractals	T-S fuzzy-NN	AFSMC based on fuzzy Lyapunov synthesis	Improved version of Adams-Bashforth-Moulton algorithm	[74]
2011	USA	IEEE Conference	FOHNN	NN with ring structure	Adams-Bashforth-Moulton predictor-corrector	[35]
2011	Poland	IEEE Conference	FONN	Fuzzy Lyapunov synthesis approach	Non-linear discrete FONN	[17]
2011	Pakistan	International Journal of Innovative Computing, Information and Control	FFNN	Swarm intelligence technique	Continuous standard PSO	[11]

properties of quaternion and Caputo derivatives. Further, the global asymptotical stability condition, comparison theory, and global synchronization for FONNs with multiple time delays have been reported [89]. Here, measures like memristor-based anti-synchronization control have been applied. In [22], sufficient conditions have been obtained on the asymptotic stability for Riemann-Liouville FONNs with discrete and distributed delays. Here, the Lipschitz continuous functions have been used as activation functions. Similar analysis on the NNs with time delays for their dissipativity has been reported in [73]. Here, the fractional Halanay inequality has been used to obtain new algebraic conditions. This new Lyapunov function guarantees the global asymptotic stability of FVCNNs with time delays.

In [40], the paper investigates the global Mittag-Leffler stability for FOHNN with impulsive effects. Here, the *tanh* function has been used as the activation function. The global synchronization for FOMNN with parameter uncertainty has been obtained in [55]. In [29], the paper aimed to establish a framework for the fundamental stability analysis of the FONNs. Here, the authors worked towards relaxing the knowledge requirements of system uncertainties in the controller design. For activation functions, here, a *tanh*-non-linear function has been used.

On the other hand, the work reported in [90] compares and studies the linear delayed systems' stability. Lipschitz continuous functions have been used here as activation functions. The work in [21] discusses the class of delayed FONNs finite-time stability. Here, two effective criteria were derived to ensure the finite-time stability of fractional-order systems. For activation function, *tanh* has been used here. The uniform stability analysis conditions for FOCNNs with time delays have been reported in [72]. The paper [71] investigates the stability of memristor-based FOCVNN with time delays. CNN's have complex-valued states, connection weights, and activation functions. In [53], the paper studies the dynamic behavior of the class of FOMNN.

The theoretical stability analysis for FOHNNs with time delays has been reported in [38] for two FOHNNs with ring structures with time delays. Also, the work derived the corresponding sufficient conditions for the system's stability. The stability analysis of FOBAM with delays is reported in [52]. The investigation is based on the inequality technique. Also, Lipschitz continuous functions have been used as activation functions. In [19], the paper presents a sufficient condition for a class of FONNs with time delay has been given to ensure the networks' uniform stability. Here, piecewise linear and Lipschitz continuous functions have been used as activation functions. The finite-time stability for a class of FONNs with time delays has been reported in [20]. The analysis helps in the design and application of fractional networks. The work in [37] synchronizes a class of chaotic FONNs by using Mittag-Leffler functions. Here, Lipschitz continuous function has been used as an activation function. The theoretical stability analyses of FOHNN in [36] reported that the FONNs exhibit chaotic dynamic behavior. A unified form is first established in [92] for fractional-order multidimensional-valued BAM neural networks. A criterion is derived from new LKFs in a vector form and two new inequalities. This has shown higher flexibility, more negligible computation, more diversity, and lower conservatism. The work presented in [93] has proposed a proper LKF by using the differential inclusions to achieve global asymptotic stability of fractional-order uncertain BAM competitive neural networks. This is based on the LMI technique and has shown the feasibility and effectiveness of this method.

A novel fractional-order differential inequality has been introduced, i.e., the α-exponential stability in [18]. Also, some effective criteria have been derived for such kind of stability. Here, Lipschitz and *tanh* functions have been used for activation functions. A novel approach using fuzzy and sliding models has been proposed to synchronize the fractional-order Duffing-Holmes chaotic system [74]. The theoretical stability analysis for the FOHNN has been presented in [35]. The proposed study is the first attempt towards theoretically characterizing the Hopfield bifurcation phenomena in FONN. The *tanh* function has been used here as the activation function.

3.3. Optimization Problems

In 2011, an attempt to solve Fractional-order differential equations using NN was made using the swarm intelligence technique and *log sigmoid* function as activation function. Statistical data such as fitness values, best, worst, standard deviation are used as performance measures [11]. In 2016 Shuo Zhang et al. analyzed the dynamic properties of the FOMNN system's global attractivity. Firstly, a growth condition is presented. Later, the global Mittag-Leffler stability for a FOMNN system is analyzed. In 2020, solving the Fractional-order partial differential equations using NN is proposed. The analysis has been applied in various problems such as fractional wave equations, boundary value problems, fractional heat conduction equations [12]. In the same year, an approach using NN in solving the delay fractional optimal control problems by an indirect method is presented. Suitable transfer functions are used as activation functions [13, 94, 95, 96, 97]. Within the year 2021, a way to numerically approximate the initial and boundary value problems by using the Haar wavelets technique has been proposed. The proposed method is applied to linear and non-linear fractional-order differential equations, and numerical results show that the proposed method is efficient [84]. Later, an efficient fractional global learning machine called Fragmachine was developed to find a model's global optimal solution and determine the optimal search path [16]. Also, a quantum backpropagation algorithm based on the fractional Grünwald–Letnikov is proposed in [15] to reduce the convergence speed and error.

3.4. Communication Systems

In 2012, the researchers investigated the hyperchaotic phenomena in the newly proposed FOCNN model in the application of secure communication to improve the system's security [42]. In 2013 Xia Huang et al. investigated complex dynamics of FOHNN with time delays. They discovered a wide variety of interesting dynamic behavior of the system, such as periodic and chaotic [91]. In 2016 the global projective synchronization of non-identical FONN based on sliding mode control technique was studied, and sufficient criteria were derived [27]. In 2017 an attempt to explore the Fractional-order application of a FONN system synchronization in the areas of secure communication was made. Also, they created an integral sliding surface model, and the generalized projective synchronization of the FONN system with time delays is explored [28].

4. Conclusion

The advances made in the field of fractional-order neural networks along with the development of their structures, learning algorithm, methods, dynamics in the last decade are discussed and summarized in this paper. Various forms of fractional-order neural networks such as FFNN, FOHNN, FOCNN, FOMNN, FOCVNN, and other networks are analyzed critically. Also, a review of applications of these neural networks in system identification and control, stability analysis and synchronization, optimization, and communication is presented.

Acknowledgment

The authors are thankful for the research supported by the Vellore Institute of Technology and Second Century Fund (C2F), Chulalongkorn University.

References

[1] C. Huang, H. Liu, X. Shi, X. Chen, M. Xiao, Z. Wang, and J. Cao, "Bifurcations in a fractional-order neural network with multiple leakage delays," *Neural Networks*, vol. 131, pp. 115–126, 2020.

[2] K. Bingi, R. Ibrahim, M. N. Karsiti, S. M. Hassan, V. R. Harindran *et al.*, *Fractional-order systems and PID controllers*. Springer, 2020.

[3] K. Bingi, P. A. M. Devan, and B. R. Prusty, "Design and analysis of fractional filters with complex orders," in *2020 3rd International Conference on Energy, Power and Environment: Towards Clean Energy Technologies*. IEEE, 2021, pp. 1–6.

[4] K. Bingi, R. Ibrahim, M. N. Karsiti, S. M. Hassan, and V. R. Harindran, "Fractional order pi controllers for real-time control of pressure plant," in *2018 5th International Conference on Control, Decision and Information Technologies (CoDIT)*. IEEE, 2018, pp. 972–977.

[5] M. S. Job, P. H. Bhateja, M. Gupta, K. Bingi, and B. R. Prusty, "Fractional Rectified Linear Unit Activation Function and Its Variants," *Mathematical Problems in Engineering*, vol. 2022, p. 1860779, Jun. 2022, publisher: Hindawi. [Online]. Available: https://doi.org/10.1155/2022/1860779

[6] K. Bingi and B. R. Prusty, "Forecasting models for chaotic fractional–order oscillators using neural networks," *International Journal of Applied Mathematics and Computer Science*, vol. 31, no. 3, pp. 387–398, 2021.

[7] K. Bingi, B. R. Prusty, A. Kumra, and A. Chawla, "Torque and temperature prediction for permanent magnet synchronous motor using neural networks," in *2020 3rd International Conference on Energy, Power and Environment: Towards Clean Energy Technologies*. IEEE, 2021, pp. 1–6.

[8] K. Bingi and B. R. Prusty, "Chaotic time series prediction model for fractional-order duffing's oscillator," in *2021 8th International Conference on Smart Computing and Communications (ICSCC)*. IEEE, 2021, pp. 357–361.

[9] N. B. Shaik, S. R. Pedapati, A. Othman, K. Bingi, and F. A. Abd Dzubir, "An intelligent model to predict the life condition of crude oil pipelines using artificial neural networks," *Neural Computing and Applications*, pp. 1–22, 2021.

[10] K. Bingi, P. A. M. Devan, and F. A. Hussin, "Reconstruction of chaotic attractor for fractional-order tamaševičius system using recurrent neural networks," in *2021 Australian & New Zealand Control Conference (ANZCC)*. IEEE, 2021, pp. 1–6.

[11] M. A. Z. Raja, J. Khan, and I. Qureshi, "Swarm intelligent optimized neural networks for solving fractional differential equations," *International Journal of Innovative Computing, Information and Control*, vol. 7, no. 11, pp. 6301–6318, 2011.

[12] H. Qu, X. Liu, and Z. She, "Neural network method for fractional-order partial differential equations," *Neurocomputing*, vol. 414, pp. 225–237, 2020.

[13] F. Kheyrinataj and A. Nazemi, "Fractional power series neural network for solving delay fractional optimal control problems," *Connection Science*, vol. 32, no. 1, pp. 53–80, 2020.

[14] P.-Y. Xiong, H. Jahanshahi, R. Alcaraz, Y.-M. Chu, J. Gómez-Aguilar, and F. E. Alsaadi, "Spectral entropy analysis and synchronization of a multi-stable fractional-order chaotic system using a novel neural network-based chattering-free sliding mode technique," *Chaos, Solitons & Fractals*, vol. 144, p. 110576, 2021.

[15] Y. Dong, X. Li, J. Zhang, Z. Li, and D. Hou, "Application of fractional theory in quantum back propagation neural network," *Mathematical Methods in the Applied Sciences*, 2021.

[16] H. Zhang, Y.-F. Pu, X. Xie, B. Zhang, J. Wang, and T. Huang, "A global neural network learning machine: Coupled integer and fractional calculus operator with an adaptive learning scheme," *Neural Networks*, vol. 143, pp. 386–399, 2021.

[17] D. Sierociuk and I. Petráš, "Modeling of heat transfer process by using discrete fractional-order neural networks," in *2011 16th International Conference on Methods & Models in Automation & Robotics*. IEEE, 2011, pp. 146–150.

[18] J. Yu, C. Hu, and H. Jiang, "α-stability and α-synchronization for fractional-order neural networks," *Neural Networks*, vol. 35, pp. 82–87, 2012.

[19] L. Chen, Y. Chai, R. Wu, T. Ma, and H. Zhai, "Dynamic analysis of a class of fractional-order neural networks with delay," *Neurocomputing*, vol. 111, pp. 190–194, 2013.

[20] W. Ran-Chao, H. Xin-Dong, and C. Li-Ping, "Finite-time stability of fractional-order neural networks with delay," *Communications in Theoretical Physics*, vol. 60, no. 2, p. 189, 2013.

[21] L. Chen, C. Liu, R. Wu, Y. He, and Y. Chai, "Finite-time stability criteria for a class of fractional-order neural networks with delay," *Neural Computing and Applications*, vol. 27, no. 3, pp. 549–556, 2016.

[22] H. Zhang, R. Ye, S. Liu, J. Cao, A. Alsaedi, and X. Li, "Lmi-based approach to stability analysis for fractional-order neural networks with discrete and distributed delays," *International Journal of Systems Science*, vol. 49, no. 3, pp. 537–545, 2018.

[23] Y. Yang, Y. He, Y. Wang, and M. Wu, "Stability analysis of fractional-order neural networks: An lmi approach," *Neurocomputing*, vol. 285, pp. 82–93, 2018.

[24] C. Huang and J. Cao, "Bifurcation mechanisation of a fractional-order neural network with unequal delays," *Neural Processing Letters*, vol. 52, no. 2, pp. 1171–1187, 2020.

[25] W. Ma, C. Li, Y. Wu, and Y. Wu, "Adaptive synchronization of fractional neural networks with unknown parameters and time delays," *Entropy*, vol. 16, no. 12, pp. 6286–6299, 2014.

[26] J. Yu, C. Hu, H. Jiang, and X. Fan, "Projective synchronization for fractional neural networks," *Neural Networks*, vol. 49, pp. 87–95, 2014.

[27] Z. Ding and Y. Shen, "Projective synchronization of nonidentical fractional-order neural networks based on sliding mode controller," *Neural Networks*, vol. 76, pp. 97–105, 2016.

[28] X. Song, S. Song, B. Li, and I. Tejado Balsera, "Adaptive projective synchronization for time-delayed fractional-order neural networks with uncertain parameters and its application in secure communications," *Transactions of the Institute of Measurement and Control*, vol. 40, no. 10, pp. 3078–3087, 2018.

[29] H. Liu, S. Li, H. Wang, Y. Huo, and J. Luo, "Adaptive synchronization for a class of uncertain fractional-order neural networks," *Entropy*, vol. 17, no. 10, pp. 7185–7200, 2015.

[30] Z. Ding, Y. Shen, and L. Wang, "Global mittag-leffler synchronization of fractional-order neural networks with discontinuous activations," *Neural Networks*, vol. 73, pp. 77–85, 2016.

[31] C. Song and J. Cao, "Dynamics in fractional-order neural networks," *Neurocomputing*, vol. 142, pp. 494–498, 2014.

[32] S. Zhang, Y. Yu, and J. Yu, "Lmi conditions for global stability of fractional-order neural networks," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 28, no. 10, pp. 2423–2433, 2016.

[33] Y. Gu, Y. Yu, and H. Wang, "Synchronization-based parameter estimation of fractional-order neural networks," *Physica A: Statistical Mechanics and its Applications*, vol. 483, pp. 351–361, 2017.

[34] A. Boroomand and M. B. Menhaj, "Fractional-order hopfield neural networks," in *International Conference on Neural Information Processing*. Springer, 2008, pp. 883–890.

[35] E. Kaslik and S. Sivasundaram, "Dynamics of fractional-order neural networks," in *The 2011 International Joint Conference on Neural Networks*. IEEE, 2011, pp. 611–618.

[36] ——, "Nonlinear dynamics and chaos in fractional-order neural networks," *Neural Networks*, vol. 32, pp. 245–256, 2012.

[37] L. Chen, J. Qu, Y. Chai, R. Wu, and G. Qi, "Synchronization of a class of fractional-order chaotic neural networks," *Entropy*, vol. 15, no. 8, pp. 3265–3276, 2013.

[38] H. Wang, Y. Yu, and G. Wen, "Stability analysis of fractional-order hopfield neural networks with time delays," *Neural Networks*, vol. 55, pp. 98–109, 2014.

[39] S. Zhang, Y. Yu, and Q. Wang, "Stability analysis of fractional-order hopfield neural networks with discontinuous activation functions," *Neurocomputing*, vol. 171, pp. 1075–1084, 2016.

[40] H. Wu, X. Zhang, S. Xue, L. Wang, and Y. Wang, "Lmi conditions to global mittag-leffler stability of fractional-order neural networks with impulses," *Neurocomputing*, vol. 193, pp. 148–154, 2016.

[41] I. Petras, "A note on the fractional-order cellular neural networks," in *The 2006 IEEE International Joint Con-*

ference on Neural Network Proceedings. IEEE, 2006, pp. 1021–1024.

[42] X. Huang, Z. Zhao, Z. Wang, and Y. Li, "Chaos and hyperchaos in fractional-order cellular neural networks," *Neurocomputing*, vol. 94, pp. 13–21, 2012.

[43] I. Stamova, "Global mittag-leffler stability and synchronization of impulsive fractional-order neural networks with time-varying delays," *Nonlinear Dynamics*, vol. 77, no. 4, pp. 1251–1260, 2014.

[44] T. Hu, X. Zhang, and S. Zhong, "Global asymptotic synchronization of nonidentical fractional-order neural networks," *Neurocomputing*, vol. 313, pp. 39–46, 2018.

[45] A. Pratap, R. Raja, R. Agarwal, J. Cao, O. Bagdasar *et al.*, "Multi-weighted complex structure on fractional order coupled neural networks with linear coupling delay: a robust synchronization problem," 2020.

[46] H. Qu, T. Zhang, and J. Zhou, "Global stability analysis of s-asymptotically ω-periodic oscillation in fractional-order cellular neural networks with time variable delays," *Neurocomputing*, vol. 399, pp. 390–398, 2020.

[47] J. Xiao, Y. Li, and S. Wen, "Mittag-leffler synchronization and stability analysis for neural networks in the fractional-order multi-dimension field," *Knowledge-Based Systems*, vol. 231, p. 107404, 2021.

[48] M. Hui, C. Wei, J. Zhang, H. Ho-Ching Iu, N. Luo, R. Yao, and L. Bai, "Finite-time projective synchronization of fractional-order memristive neural networks with mixed time-varying delays," *Complexity*, vol. 2020, 2020.

[49] W. Sun, G. Ren, Y. Yu, and X. Hai, "Global synchronization of reaction-diffusion fractional-order memristive neural networks with time delay and unknown parameters," *Complexity*, vol. 2020, 2020.

[50] A. Pratap, R. Raja, J. Alzabut, J. Dianavinnarasi, J. Cao, and G. Rajchakit, "Finite-time mittag-leffler stability of fractional-order quaternion-valued memristive neural networks with impulses," *Neural Processing Letters*, vol. 51, no. 2, pp. 1485–1526, 2020.

[51] X. Huang, J. Jia, Y. Fan, Z. Wang, and J. Xia, "Interval matrix method based synchronization criteria for fractional-order memristive neural networks with multiple time-varying delays," *Journal of the Franklin Institute*, vol. 357, no. 3, pp. 1707–1733, 2020.

[52] X. Yang, Q. Song, Y. Liu, and Z. Zhao, "Uniform stability analysis of fractional-order bam neural networks with delays in the leakage terms," in *Abstract and applied analysis*, vol. 2014. Hindawi, 2014.

[53] J. Chen, Z. Zeng, and P. Jiang, "Global mittag-leffler stability and synchronization of memristor-based fractional-order neural networks," *Neural Networks*, vol. 51, pp. 1–8, 2014.

[54] Y. Cao and C. Bai, "Finite-time stability of fractional-order bam neural networks with distributed delay," in *Abstract and Applied Analysis*, vol. 2014. Hindawi, 2014.

[55] Y. Gu, Y. Yu, and H. Wang, "Synchronization for fractional-order time-delayed memristor-based neural networks with parameter uncertainty," *Journal of the Franklin Institute*, vol. 353, no. 15, pp. 3657–3684, 2016.

[56] S. Zhang, Y. Yu, and Y. Gu, "Global attractivity of memristor-based fractional-order neural networks," *Neurocomputing*, vol. 227, pp. 64–73, 2017.

[57] X. Yang, C. Li, T. Huang, Q. Song, and X. Chen, "Quasi-uniform synchronization of fractional-order memristor-based neural networks with delay," *Neurocomputing*, vol. 234, pp. 205–215, 2017.

[58] A. Pratap, R. Raja, C. Sowmiya, O. Bagdasar, J. Cao, and G. Rajchakit, "Global projective lag synchronization of fractional order memristor based bam neural networks with mixed time varying delays," *Asian Journal of Control*, vol. 22, no. 1, pp. 570–583, 2020.

[59] H.-L. Li, H. Jiang, and J. Cao, "Global synchronization of fractional-order quaternion-valued neural networks with leakage and discrete delays," *Neurocomputing*, vol. 385, pp. 211–219, 2020.

[60] J. Xiao, J. Cao, J. Cheng, S. Zhong, and S. Wen, "Novel methods to finite-time mittag-leffler synchronization problem of fractional-order quaternion-valued neural networks," *Information Sciences*, vol. 526, pp. 221–244, 2020.

[61] G. Narayanan, M. S. Ali, M. I. Alam, G. Rajchakit, N. Boonsatit, P. Kumar, and P. Hammachukiattikul, "Adaptive fuzzy feedback controller design for finite-time mittag-leffler synchronization of fractional-order quaternion-valued reaction-diffusion fuzzy molecular modeling of delayed neural networks," *IEEE Access*, vol. 9, pp. 130 862–130 883, 2021.

[62] A. Pratap, R. Raja, J. Alzabut, J. Cao, G. Rajchakit, and C. Huang, "Mittag-leffler stability and adaptive impulsive synchronization of fractional order neural networks in quaternion field," *Mathematical Methods in the Applied Sciences*, vol. 43, no. 10, pp. 6223–6253, 2020.

[63] U. Humphries, G. Rajchakit, P. Kaewmesri, P. Chanthorn, R. Sriraman, R. Samidurai, and C. P. Lim, "Global stability analysis of fractional-order quaternion-valued bidirectional associative memory neural networks," *Mathematics*, vol. 8, no. 5, p. 801, 2020.

[64] Y. Gu, H. Wang, and Y. Yu, "Synchronization for commensurate riemann-liouville fractional-order memristor-based neural networks with unknown parameters," *Journal of the Franklin Institute*, vol. 357, no. 13, pp. 8870–8898, 2020.

[65] J. Xiao, J. Cheng, K. Shi, and R. Zhang, "A general approach to fixed-time synchronization problem for

fractional-order multidimension-valued fuzzy neural networks based on memristor," *IEEE Transactions on fuzzy systems*, vol. 30, no. 4, pp. 968–977, 2021.

[66] J. Xiao, J. Cao, J. Cheng, S. Wen, R. Zhang, and S. Zhong, "Novel inequalities to global mittag–leffler synchronization and stability analysis of fractional-order quaternion-valued neural networks," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 32, no. 8, pp. 3700–3709, 2020.

[67] J. Xiao, S. Zhong, and S. Wen, "Unified analysis on the global dissipativity and stability of fractional-order multidimension-valued memristive neural networks with time delay," *IEEE Transactions on Neural Networks and Learning Systems*, 2021.

[68] G. Rajchakit, P. Chanthorn, M. Niezabitowski, R. Raja, D. Baleanu, and A. Pratap, "Impulsive effects on stability and passivity analysis of memristor-based fractional-order competitive neural networks," *Neurocomputing*, vol. 417, pp. 290–301, 2020.

[69] A. Pratap, R. Raja, J. Alzabut, J. Dianavinnarasi, J. Cao, and G. Rajchakit, "Finite-time mittag-leffler stability of fractional-order quaternion-valued memristive neural networks with impulses," *Neural Processing Letters*, vol. 51, no. 2, pp. 1485–1526, 2020.

[70] G. Rajchakit, P. Chanthorn, P. Kaewmesri, R. Sriraman, and C. P. Lim, "Global mittag–leffler stability and stabilization analysis of fractional-order quaternion-valued memristive neural networks," *Mathematics*, vol. 8, no. 3, p. 422, 2020.

[71] R. Rakkiyappan, G. Velmurugan, and J. Cao, "Finite-time stability analysis of fractional-order complex-valued memristor-based neural networks with time delays," *Nonlinear Dynamics*, vol. 78, no. 4, pp. 2823–2836, 2014.

[72] R. Rakkiyappan, J. Cao, and G. Velmurugan, "Existence and uniform stability analysis of fractional-order complex-valued neural networks with time delays," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 26, no. 1, pp. 84–97, 2014.

[73] G. Velmurugan, R. Rakkiyappan, V. Vembarasan, J. Cao, and A. Alsaedi, "Dissipativity and stability analysis of fractional-order complex-valued neural networks with time delay," *Neural Networks*, vol. 86, pp. 42–53, 2017.

[74] T.-C. Lin, T.-Y. Lee, and V. E. Balas, "Adaptive fuzzy sliding mode control for synchronization of uncertain fractional order chaotic systems," *Chaos, Solitons & Fractals*, vol. 44, no. 10, pp. 791–801, 2011.

[75] J. Fei, Z. Wang, X. Liang, Z. Feng, and Y. Xue, "Fractional sliding mode control for micro gyroscope based on multilayer recurrent fuzzy neural network," *IEEE Transactions on Fuzzy Systems*, 2021.

[76] A. P. Singh, D. Deb, H. Agrawal, K. Bingi, and S. Ozana, "Modeling and control of robotic manipulators: A fractional calculus point of view," *Arabian Journal for Science and Engineering*, pp. 1–12, 2021.

[77] X. Wang, J. Wang, and M. Fečkan, "Bp neural network calculus in economic growth modelling of the group of seven," *Mathematics*, vol. 8, no. 1, p. 37, 2020.

[78] H. Wang, "Research on application of fractional calculus in signal real-time analysis and processing in stock financial market," *Chaos, Solitons & Fractals*, vol. 128, pp. 92–97, 2019.

[79] H. Jahanbakhti, "A novel fractional-order neural network for model reduction of large-scale systems with fractional-order nonlinear structure," *Soft Computing*, pp. 1–11, 2020.

[80] J. Chaudhary, H. Puri, R. Mantri, K. R. Raghavendra, and K. Bingi, "Prediction of turbidity in beach waves using nonlinear autoregressive neural networks," in *2021 8th International Conference on Smart Computing and Communications (ICSCC)*. IEEE, 2021, pp. 42–46.

[81] J. Zamora Esquivel, A. Cruz Vargas, R. Camacho Perez, P. Lopez Meyer, H. Cordourier, and O. Tickoo, "Adaptive activation functions using fractional calculus," in *Proceedings of the IEEE/CVF International Conference on Computer Vision Workshops*, 2019, pp. 0–0.

[82] F. Marques, T. P. De Araujo, C. Nator, A. Saraiva, J. Sousa, A. M. Pinto, and R. Melo, "Recognition of simple handwritten polynomials using segmentation with fractional calculus and convolutional neural networks," in *2019 8th Brazilian Conference on Intelligent Systems (BRACIS)*. IEEE, 2019, pp. 245–250.

[83] P. P. Mehta, G. Pang, F. Song, and G. E. Karniadakis, "Discovering a universal variable-order fractional model for turbulent couette flow using a physics-informed neural network," *Fractional Calculus and Applied Analysis*, vol. 22, no. 6, pp. 1675–1688, 2019.

[84] V. Mehandiratta, M. Mehra, and G. Leugering, "An approach based on haar wavelet for the approximation of fractional calculus with application to initial and boundary value problems," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 4, pp. 3195–3213, 2021.

[85] C. Z. Aguilar, J. Gómez-Aguilar, V. Alvarado-Martínez, and H. Romero-Ugalde, "Fractional order neural networks for system identification," *Chaos, Solitons & Fractals*, vol. 130, p. 109444, 2020.

[86] L. Wang, Y. Xie, X. Wang, J. Xu, H. Zhang, J. Yu, Q. Sun, and Z. Zhao, "Meteorological sequence prediction based on multivariate space-time auto regression model and fractional calculus grey model," *Chaos, Solitons & Fractals*, vol. 128, pp. 203–209, 2019.

[87] P. A. Schirmer and I. Mporas, "Energy disaggregation using fractional calculus," in *ICASSP 2020-2020*

IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP). IEEE, 2020, pp. 3257–3261.

[88] C. Huang, X. Zhao, X. Wang, Z. Wang, M. Xiao, and J. Cao, "Disparate delays-induced bifurcations in a fractional-order neural network," *Journal of the Franklin Institute*, vol. 356, no. 5, pp. 2825–2846, 2019.

[89] Y. Gu, Y. Yu, and H. Wang, "Synchronization for fractional-order time-delayed memristor-based neural networks with parameter uncertainty," *Journal of the Franklin Institute*, vol. 353, no. 15, pp. 3657–3684, 2016.

[90] S. Liang, R. Wu, and L. Chen, "Comparison principles and stability of nonlinear fractional-order cellular neural networks with multiple time delays," *Neurocomputing*, vol. 168, pp. 618–625, 2015.

[91] X. Huang, Z. Wang, and Y. Li, "Nonlinear dynamics and chaos in fractional-order hopfield neural networks with delay," *Advances in Mathematical Physics*, vol. 2013, 2013.

[92] J. Xiao, S. Zhong, and S. Wen, "Improved approach to the problem of the global mittag-leffler synchronization for fractional-order multidimension-valued bam neural networks based on new inequalities," *Neural Networks*, vol. 133, pp. 87–100, 2021.

[93] M. Syed Ali, M. Hymavathi, S. A. Kauser, G. Rajchakit, P. Hammachukiattikul, and N. Boonsatit, "Synchronization of fractional order uncertain bam competitive neural networks," *Fractal and Fractional*, vol. 6, no. 1, p. 14, 2021.

[94] H. Puri, J. Chaudhary, K. R. Raghavendra, R. Mantri, and K. Bingi, "Prediction of heart stroke using support vector machine algorithm," in *2021 8th International Conference on Smart Computing and Communications (ICSCC)*. IEEE, 2021, pp. 21–26.

[95] R. Mantri, K. R. Raghavendra, H. Puri, J. Chaudhary, and K. Bingi, "Weather prediction and classification using neural networks and k-nearest neighbors," in *2021 8th International Conference on Smart Computing and Communications (ICSCC)*. IEEE, 2021, pp. 263–268.

[96] N. B. Shaik, M. Inayat, W. Benjapolakul, B. Bakthavatchalam, S. D. Barewar, W. Asdornwised, and S. Chaitusaney, "Artificial neural network modeling and optimiztion of thermophysical behavior of mxene ionanofluids for hybrid solar photovoltaic and thermal systems," *Thermal Science and Engineering Progress*, p. 101391, 2022.

[97] N. B. Shaik, W. Benjapolakul, S. R. Pedapati, K. Bingi, N. T. Le, W. Asdornwised, and S. Chaitusaney, "Recurrent neural network-based model for estimating the life condition of a dry gas pipeline," *Process Safety and Environmental Protection*, vol. 164, pp. 639–650, 2022.

Monalisa Maiti completed her higher secondary education from Air Force School Viman Nagar, Pune. She is originally from West Bengal, India but has lived across different parts of India. Currently, she is pursuing her Bachelor's degree (BTech) in Electronics and Instrumentation Engineering at Vellore Institute of Technology, Vellore. Her research interests centre around the development and application of a Hybrid Optimization Algorithm. She is a part of the management department in the team Assailing Falcons and Skillship Vellore chapter. She is an aeroplane enthusiast and has a penchant for English Literature.

Sunder M comes from Tamil Nadu, India completed his higher secondary education in 2019 and is currently pursuing his Bachelor's degree (B.Tech) in the Department of Electronics and Instrumentation Engineering, School of Electrical Engineering at Vellore Institute of Technology, Vellore. His current research interest includes Hybrid Optimization Algorithm. He has successfully co-prototyped All-Terrain-Rover guided by a small synchronized rotorcraft. For his passion on 3D Animation and Photorealistic physics simulations, he contributes to his University's Animation team as an Animation Artist.

DOI:10.4186/ej.2022.26.7.49

Abishek R from Tamil Nadu, India graduated with flying colours and completed his higher secondary education from Maharishi Vidya Mandir Sr.Sec School, Hosur and is currently pursuing his Bachelor's degree (B.Tech) in the Department of Electronics and Instrumentation Engineering, School of Electrical Engineering at VIT University, Vellore. His current research interest includes Optimization Algorithm. He is also a key contributor of the design team in his University. His interests include photography and sports, in which he has competed and won prizes in both. Being part of NCC in high school has also honed discipline and expert marksmanship skills in him.

Kishore Bingi currently works as an assistant professor the Department of Control and Automation, School of Electrical Engineering, Vellore Institute of Technology, Vellore, India. Currently, he is holding the position of Inch Committee and Research Expert Committee members at the school. He obtained his PhD degree in the Department of Electrical and Electronic Engineering, Universiti Teknologi PETRONAS, Perak, Malaysia, in 2019, his M.Tech degree in instrumentation and control systems from the National Institute of Technology, Calicut, Kerala, India, in 2014, and his BTech degree in electrical and electronics engineering from Bapatla Engineering College, Andhra Pradesh, India, in 2012. He worked as a research scientist and postdoctoral researcher in the Institute of Autonomous Systems, Universiti Teknologi PETRONAS, from 2019 to 2020. He also worked with TATA Consultancy Service as an assistant systems engineer from 2014 to 2015. He served as Chair of IEEE UTP Student Branch from Jan 2018 - Dec 2019. Currently he is supervising 2 Ph.D. scholars and 10 undergraduates for their Capstone projects. He published over 3 books, 23 journals, 40 conferences, 6 book chapters and 4 copyrights from Malaysian IPO. He has won several awards including, silver medal in expos, best paper presentations, international travel grants etc. His current research interests include chaos prediction, non-linear process modeling, fractional-order control, and optimization.

Nagoor Basha Shaik is a Post-Doctoral Researcher in Artificial Intelligence (A.I.) and Machine Learning (ML) and Smart Grid Technology Research Unit, Faculty of Engineering, from Chulalongkorn University. He did his Ph.D. in the Department of Mechanical Engineering, specializing in A.I. and ML applications - in the oil & gas sector from Universiti Teknologi PETRONAS, Malaysia (2017-2021). He finished his M. Tech in the Department of Mechanical Engineering (Specialised in Machine Design) from Jawaharlal Nehru Technological University, Kakinada, India, in 2016. He completed his Bachelor of Technology in Mechanical Engineering from Acharya Nagarjuna University, Guntur, A.P in 2011. He worked as a mechanical engineer in India from July 2011- Nov 2014 in India. He published more than 20 articles in good impact factor journals. His area of interest includes materials, artificial intelligence, oil and gas pipelines, piping systems, prediction techniques, manufacturing processes, engineering drawing, and material properties.

Watit Benjapolakul received a D.Eng. degree in electrical engineering from The University of Tokyo in 1989. He is currently a Professor with the Department of Electrical Engineering, Chulalongkorn University, Thailand. He is also the Head of the Artificial Intelligence, Machine Learning, and Smart Grid Technology Research Unit, Faculty of Engineering, Chulalongkorn University. His current research interests include mobile communication systems, broadband networks, applications of artificial intelligence in communication systems, wireless networks, and applications of communication networks in smart grids.