Review

Recent Advances and Applications of Fractional-Order Neural Networks

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Abstract. This paper focuses on the growth, development, and future of various forms of fractional-order neural networks. Multiple advances in structure, learning algorithms, and methods have been critically investigated and summarized. This also includes the recent trends in the dynamics of various fractional-order neural networks. The multiple forms of fractional-order neural networks considered in this study are Hopfield, cellular, memristive, complex, and quaternion-valued based networks. Further, the application of fractional-order neural networks in various computational fields such as system identification, control, optimization, and stability have been critically analyzed and discussed.

Keywords: Cellular networks, control systems, fractional calculus, Hopfield networks, identification, memristive neural networks.

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Nomenclature

BAM Bidirectional Associative Memory
CMOS Complementary Metal-Oxide Semiconductor
CNN Cellular Neural Network
CVNN Complex Valued Neural Networks
FFNN Feed Forward Neural Network
FOBAM Fractional-order BAM Neural Network
FOCNN Fractional-order Cellular Neural Network
FOCVNN Fractional-order Complex Valued Neural Networks
FOHNN Fractional-order Hopfield Neural Networks
FOMNN Fractional-order Memristive Neural Networks
FONN Fractional-order Neural Network
FOQVNN Fractional-order Quaternion-Valued Neural Networks
FOQVNNLDD Fractional-order Quaternion-Valued Neural Networks with Leakage and Discrete Delays
FOQVMNN Fractional-order Quaternion-Valued Memristive Neural Networks
FOPBAM Fractional-order Quaternion-Valued Bidirectional Associative Memory
LKF Lyapunov-Krasovskii functional
LMI Linear Matrix Inequality
MFOBNN Memristor-based Fractional-order BAM Neural Network
MVM Matrix-Vector Multiplication
NARX Nonlinear Auto-regressive Exogenous Network
NN Neural Network
PSO Particle Swarm Optimization
QVNN Quaternion-Valued Neural Networks
ReLU Rectified Linear Unit

1. Introduction

In recent years it has been proven that combining fractional calculus and neural networks will give higher efficiency in the overall model [1]. Compared to conventional NN, the FONN describes the properties of the neurons such as memory and hereditary of various systems effectively. This effectiveness is because the fractional-order systems have infinite memory [2, 3, 4]. Also, it is to highlight that the computation ability of FONN gives efficient data processing, stimulus anticipation, and phase shift during the oscillatory neurons firing. Therefore, the fractional-order form of neural networks has been an excellent and powerful tool in various computational fields. Recently, the authors of [5] have proposed the fractional-order form of ReLU activation functions and its variants. The developed activation functions have shown better performance compared to conventional. Thus, this study provides a critical review of recent advances in neural network structures such as Hopfield, cellular, memristor-based, complex-valued, etc. Also, a review of applications of these networks for system identification, control, stability, optimization, etc., is presented.

The remaining sections of the paper are organized as follows: Section 2 presents the recent advances in FONNs. The application of FONNs in various fields is described in Section 3. Finally, the conclusions and future directions are given in Section 4.

2. Advances in Fractional-Order Neural Networks

This section presents the critical review on various advances in the fractional-order form of Hopfield, Cellular, Memristive, Complex-Valued, and other neural networks.

2.1. Feed Forward Neural Networks

FFNN is one of the most straightforward variants of neural networks as they pass information in one direction through various input nodes until it makes it to the output node [6, 7, 8]. On the other hand, RNN is more complex than FFNN. Here, each node in the RNN acts as a memory cell, continuing the computation and implementation of operations [9]. The networks save the output of processing nodes and feed them back into the model, and hence they did not pass the information in one direction only [10].

In 2011, the researchers developed an FFNN with PSO to solve differential equations [11]. Later, the work in [12] established FFNN based on sine and cosine, which helped solve fractional wave and boundary value problems. In 2020, the researchers considered fractional derivatives based on Caputo’s derivative to develop a method for op-
timizing fractional-order delay optimum control problems [13]. Also, algorithms such as using Grünwald-Letnikov’s derivative function have been equipped to model an FFNN that focuses on reducing the number of parameters required for system identification. To study the dynamical behavior of a multi-stable fractional-order system, in 2021, the method of fractional-order behavior of robotic manipulators has been equipped with the wolf algorithm [14]. Also, another reported work in [15] has used Grünwald-Letnikov’s derivative function as a learning algorithm and fractional calculus to reduce convergence error and convergence speed. The single-point search algorithm has made an efficient global learning machine to determine the optimal search path [16].

2.2. Fractional-Order Neural Networks

In [17], the work deals with the discrete FONN with a Fuzzy Lyapunov synthesis approach has been reported. Here, a non-linear discrete FONN algorithm has been implemented to apply this proposed NN in modeling a heat transfer. In [18], the paper investigates a class of FONNs. Initially, \( \alpha \)-exponential stability is introduced along with a novel differential inequality-based fractional-order method for analyzing the network’s stability. The work also proposes the use of fractional ordering for chaotic network synchronization. The equilibrium point’s existence and \( \alpha \)-exponential stability have been considered based on investigations. The work in [19] discusses a class of FONNs with delay. This work establishes an adequate condition for the uniform stability of such FONNs with a delay. Also, the equilibrium point’s uniqueness, existence, and stability have been shown.

Further, in [20], the paper investigates a category of FONNs for their finite-time stabilities. It is worth highlighting that sufficient conditions have been introduced to establish the time-stability of such a class of FONNs with the Caputo fractional derivatives. These conditions are based on the Gronwall theorem, Laplace transforms, and Mittag-Leffler approximation approaches. Here, the asymptotic stability results of such FONNs are also shown. In [21], a category of FONNs with delayed systems has been scrutinized for their finite-time stabilities. It has been observed that the inequality technique is used to obtain two novel sufficient conditions that are delay-dependent, which ensures the stability of such FONNs. Also, the Caputo derivative has been derived and verified with Cauchy-Schwartz inequality. In [22], the paper focuses on the FONNs based on the Riemann-Liouville that have discrete and distributed delays and analyses their asymptotic stability. Here, two sufficient conditions are derived by constructing a suitable Lyapunov functional to ensure that the addressed NN is stable asymptotically.

In [23], stability analysis of FONNs with and without delay using fractional approaches such as Lyapunov and Razumikhin has been reported. It has been observed that the non-linear constraints were handled using the S-procedure to get a more comprehensive selection of the systems. The uniqueness and existence of the point of equilibrium have been proved in this work. Unlike the last, in [24], the article focuses on FONN with double leakage delays and analyzes their stability and bifurcation. This is done by considering leftover delay as a bifurcation parameter after intercalating one delay. It has been observed that incongruent critical values have been gained for different delays-induced bifurcations. Also, the implementation of Hopf bifurcation can be seen in this paper. In [25], the problem of parameters synchronization and identification of FONNs with time delays have been investigated. It has been observed that some analytical techniques and an adaptive control method were designed here to synchronize the two uncertain, complex networks with time delays.

Further, [26] investigates the global projective synchronization of FONNs by combining open-loop control and adaptive control. It is worth highlighting that based on Caputo’s fractional derivation, a novel fractional-order differential inequality has been derived in this paper to establish the monotonicity of the continuous and differential functions. It also applies several control strategies to ensure complete synchronization, anti-synchronization, and stabilization of the addressed NNs. Contrary to [26], in paper [27], the focus is on the non-identical FONN based on the sliding mode control technique. Here, the global projective synchronization of such FONNs is investigated. It has been observed that the authors gave derived synchronization conditions with uncertainty parameters for memristor-based NNs with fractional-order multiple time-delays that are based on Filippov’s solution and inclusion theory. In [28], it has been observed that adaptive projective synchronization of time-delayed FONNs has been analyzed. Here, efficient hybrid control strategies are designed for delayed FONNs with uncertain parameters using active and adaptive control methods. Adaptive synchronization-based method and Adams-Bashforth-Moulton predictor-corrector scheme have been implemented.

The paper in [29] studies the synchronization for a category of uncertain FONNs that have disturbed parameters. In this paper, an adaptive synchronization controller is designed based on the fractional-order extension of the Lyapunov stability criterion. Also, a fractional-order adaptation law is proposed to update the controller parameter online. The work in [30] is based on Global Mittag-Leffler’s synchronization in FONNs with discontinuous activation functions. Here, singular Gronwall inequality with properties of fractional calculus is used to prove the existence of a global solution under the framework of Filippov for FONN with discontinuous activation functions. Also, the framework of the Filippov solution for such NNs following Caputo’s fractional derivative is given. This is based on the
Lyapunov stability criterion. This paper also provided a clue to study FONNs with discontinuous functions. Also, some requirements have been established to ensure the asymptotic stability of FONNs based on the Lyapunov function’s fractional derivative. In [31], a class of NNs with fractional-order derivatives is investigated. Here, some new conditions based on Krasnoselskii’s fixed point theorem and inequality technique are employed to ensure the uniqueness and existence of the non-trivial solution. Also, the stability of the FONNs in fixed time intervals is proposed here. In [32], LMI stability conditions for linear and non-linear fractional-order systems have been presented. The global stability analysis of FONNs is employed, and the results are obtained from LMI conditions. The two-norm and fractional-order Lyapunov direct method has been implemented in this paper. Now, in [33], the article focuses on the parameter estimation problem of FONNs. Here, the identification based on the synchronization method has been generalized by applying parameter update laws and adaptive control. Also, the technique has been used for parameter identification and synchronization simultaneously.

### 2.3. Fractional-Order Hopfield Neural Networks

The architecture of the Hopfield network model is shown in Fig. 1. The model contains several linear and non-linear elements, with the size of the network being $N$. The block ‘A’ in the circuit simulates biologic neurons’ synapse and non-linear characteristics. At the same time, block ‘B’ is used to simulate the biologic neurons’ time-delay characteristics. The governing equations of the model are given as [34],

$$
C_i \frac{d^\alpha}{dt^\alpha} P_i = \sum_{j=1}^{N} \left( \frac{1}{R_{ij}} \right) V_j - P_i \left( \frac{1}{R_{i0}} + \sum_{j=1}^{N} \frac{1}{R_{ij}} \right) + I_i,
$$

and

$$
P_i = \left( \frac{1}{\lambda} \right) \varphi_i^{-1} V_i,
$$

where, $P_i$ denotes the input of operational amplifier at $i^{th}$ neuron, $V_i$ represents the output of operational amplifier at $i^{th}$ neuron, and $\lambda$ is the learning rate.

![Fig. 1. Architecture of the Hopfield network model.](image)

The governing equations for the fractional-order form of Hopfield network model is obtained from Eqs. (1) and (2) as follows:

$$
C_i \frac{d^\alpha}{dt^\alpha} P_i = \sum_{j=1}^{N} \left( \frac{1}{R_{ij}} \right) V_j - P_i \left( \frac{1}{R_{i0}} + \sum_{j=1}^{N} \frac{1}{R_{ij}} \right) + I_i,
$$

$$
P_i = \left( \frac{1}{\lambda} \right) \varphi_i^{-1} V_i,
$$

where $\frac{d^\alpha}{dt^\alpha}$ is the fractional-order derivative of order $\alpha \in (0, 1)$.

The first attempt to undertake Hopf bifurcation phenomena in FONN by theoretically characterizing has been developed in 2011 [35]. The authors have replaced fractance in the typical capacitor from the Hopfield NN, which formed a familiar FOHNN model. In 2012, they put forward a theoretical stability analysis by characteristic parameters for the FONN of Hopfield type [36]. The work observed that the FONN might show rich dynamical behavior. Their dynamics become increasingly complex as the fractional-order rises, leading to the chaotic behavior of the system. In [37], the Mittag-Leffler function has been proposed to synchronize a Class of Fractional-Order Chaotic NN. The theoretical results obtained are applied to a FOHNN model. During the same year, the complex dynamical behaviors of such networks have been inspected and observed that an expansive range of interesting dynamical phenomena has been found to exist. In 2014, to present a theoretical stability analysis of FOHNN with time delays, two FOHNN with different ring structures and delays were introduced; the derivation for the adequate conditions for stability of the systems are also provided [38]. In 2015, Shuo Zhang et al. investigated their solutions’ existence and stability conditions within the frame of Filippov solutions. Under the proposed requirements, the uniqueness of the equilibrium point of the system is broken down [39]. In 2016, the global Mittag-Leffler stability condition presented linear matrix inequalities by applying the fractional Lyapunov method with impulses [40].

### 2.4. Fractional-Order Cellular Neural Network

In CNN, the cell is the fundamental circuit element, as shown in Fig. 2. It is to note that any CNN cell is connected only to its neighbor cells. For the circuit, the input, state, and output equations are given as follows [41]:

![Image of CNN cell](image)
Fig. 2. Architecture of the cellular network model.

\[
C \frac{d}{dt} x_{ij}(t) = -\frac{1}{R_x} x_{ij}(t) + \sum_{k,l \in N_r(i,j)} A(i, j; k, l) y_{kl}(t) + \sum_{k,l \in N_r(i,j)} B(i, j; k, l) u_{kl}(t) + C(i, j; k, l) x_{kl}(t) + I,
\]

where \( C(i,j) \) is the cell of \( i^{th} \) row and \( j^{th} \) column, \( u_{ij}, x_{ij} \) and \( y_{ij} \) are the input, state and output voltages respectively of each neighbor cell \( C(k,l) \). The voltage-controlled current sources are given as,

\[
I_{xy}(i, j; k, l) = A(i, j; k, l) y_{kl},
\]

\[
I_{ux}(i, j; k, l) = B(i, j; k, l) u_{kl},
\]

\[
I_{xx}(i, j; k, l) = C(i, j; k, l) x_{kl}.
\]

In (8), (9) and (10), \( A, B \) and \( C \) are coefficients known as cloning templates. In (6) is given as,

\[
N_r(i, j) = C(k,l) : \text{max}(|k-i|, |l-j|) \leq r.
\]

From the above circuit in Fig. 2, the governing equations for the fractional-order form of CNN are given as follows:

\[
u_{ij}(t) = E_{ij},
\]

\[
C \frac{d^\alpha}{dt^\alpha} x_{ij}(t) = -\frac{1}{R_x} x_{ij}(t) + \sum_{k,l \in N_r(i,j)} A(i, j; k, l) y_{kl}(t) + \sum_{k,l \in N_r(i,j)} B(i, j; k, l) u_{kl}(t) + C(i, j; k, l) x_{kl}(t) + I,
\]

where \( \frac{d^\alpha}{dt^\alpha} \) is the fractional-order derivative of order \( \alpha \in (0,1) \).

There have been various advancements in FOCNN in techniques and learning algorithms that have potential applications. In 2012, to establish secure communications by improving the security of the chaotic communication system, FOCNN had been proposed with Fractional-order four-cell CNN [42]. Then in 2014, fractional Lyapunov method and Mittag-Leffler functions provided leverage to the development of FOCNN with time-varying delays [43]. Although in 2018, to deal with multiple-time delays and fractional-order linear delayed systems, fractional-order adaptive laws have been used to examine the stability theory in fractional-linear-delayed systems for controlling FONNs with or without sector nonlinearities [44]. As advancement in this domain, fractional-order differential inequality, including time delays, has been proposed in 2020 with a feedback controller to inspect the multi-weighted complex structure, robust synchronization of on FOCNN under linear coupling delays [45]. Researchers in 2021 have used the contraction mapping principle with FOCNN to analyze S-asymptotically \( \tau \)-periodic oscillations in FOCNN [46]. A fractional-order system of a multidimensional-valued neural network is split into four or two fractional-order systems of real-valued neural networks based on Hamilton rules. A novel inequality comprising a quadratic term is inferred and analyzed on the synchronization and stability problem [47]. Also, LKFs are constructed from these new inferred inequalities with quadratic coefficients.

2.5. Fractional-Order Memristor-Based Neural Networks

A memristor is a hardware and electrical component which performs the same function as a resistor but remembers the amount of charge that has previously flowed through it. This hardware’s specific vital properties are exploited using complex and computational intensive neuro-morphic systems such as CNNs and their derivatives. Memristors are two-terminal nano solid-state switching devices. They tend to fix the issues of memory wall and communication bottlenecks raised from the conventional CMOS technology. With the Non-Volatility property of this memristor, the capability to retain memory even without power, a memristor crossbar array architecture is proposed for practical use in real-world applications because of its high density and parallel computational ability. Memristors allow in-memory computing. They accelerate MVM, search, and bitwise operations. Most NN architectural layers include a convolutional layer whose computation pattern is MVM, and the fully-connected layer is memory-intensive. Using a memristor allows efficient hardware acceleration of CNN and other variants of NNs [48, 49, 50, 51].

In 2014, the authors proposed network stability conditions by applying an inequality approach and analysis method [52]. Further, proposed adequate requirements to ensure the equilibrium point is existing, unique, and stable. In the same year, Jiejie Chen et al. introduced memristor-based FONN, the dynamic behavior of a class of FOMNN has been proposed [53]. Moreover, sufficient criteria for
The uniqueness of the equilibrium point synchronization of the networks are also proved. In the same year, using approaches like Gronwall inequality, Laplace transforms, and estimates of Mittag-Leffler functions, the finite-time stability criterion of Fractional-Order distributed delayed BAM NN have been proposed [54]. In [55], within the frame of the Filippov solution and using differential inclusion theory, a comparison theorem has been presented for a set of fractional-order systems with multiple time-delays.

The presented requirements for attractivity for memristor-based FONN and global boundedness with non-Lipschitz activations. Further, the derivation for synchronization conditions of the system as mentioned above with parameter uncertainty is shown. Additionally, a development condition to make sure the solutions exist has been introduced by Shuo Zhang et al. [56]. In 2017, Xujun Yang et al. proposed Quasi-uniform synchronization of delayed FOMNN and several adequate criteria to ensure the quasi-uniform synchronization for the FOMNN with delay is put forward [57]. In 2018, Master-Slave synchronization for some MFBOBNN with mixed time-varying delays and switching jumps mismatch had been proposed. Some new projective lag synchronization criteria have been conferred for such networks obtained by differential inclusions theory, set-valued map, fractional Barbalat’s lemma, and control scheme [58]. In 2019, proposed a generic fractional Halanay inequality with multiple time-varying delays [51]. The delayed FOMNN is discussed by capitalizing this new upper bound and presenting inequality based on the Lyapunov function method. In the same year, Hong-Li Li et al. handled FOQVNNLDD without any disintegration [59]. The work transformed the FOQVNNs into real-valued systems, which is more concise and naturalistic than the actual decomposition method.

Two novel methods are proposed in [60] for the synchronization problem of FOQVNN. The first includes a new establishment of LKFs and fractional-order derivative inequality. The latter method equips both norm comparison rules and generalized Gronwall-Bellman inequality with the help of the Laplace transform of the Mittag-Leffler function. A Lyapunov stability theory and Caputo fractional derivative with several algebraic criteria are established in [61] for guaranteeing the finite-time Mittag-Leffler synchronization of FOQVNNs. This method has shown less conservatism than existing results. To decrease the computational complexity and avoid the non-commutativity of quaternion multiplication, FOQVNN is separated into four real-valued FONNs. Inequality techniques and Lyapunov functional criteria are proposed in [62] to ensure Mittag-Leffler stability and impulsive control. The principle of homeomorphism, the Lyapunov fractional-order method, and the matrix inequality approach are applied in [63] to derive sufficient conditions to study the global asymptotic stability problem for the FO-QVBAM neural network. The technique has shown better results confirming the existence, uniqueness, and global asymptotic stability of the system’s equilibrium point.

The global synchronization criteria are first derived by employing the second method of Lyapunov. A new set of FOQMNNs and a unique scientific expression for the quaternion-value inductance has been proposed following its features. A new sufficient criterion has been presented to verify the finite-time stability of the FOQMNN system by ignoring the impulsive effects by employing the Laplace transform in the same year. In 2020, proposed a synchronization for commensurate Riemann-Liouville FOMNN with unknown parameters through Lyapunov’s second method. Further, two inequalities of Riemann-Liouville fractional derivatives that play a vital role are provided to theoretically analyze the Riemann-Liouville fractional differential system [64]. Similarly, a fractional-based theorem has been introduced to deal with the effects of reaction-diffusion and time delay of FOMNN [49].

Additionally, the activation function has been extended to sporadic cases on the Filippov solution and set value mapping theory. In addition, a finite-time projective synchronization of FOQVNN with mixed time-varying delays and several criteria to ensure that along with the set-valued map and frame of fractional-order differential inclusion has been proposed by Meng Hui et al. [48]. Also, three properties are established, which significantly increases some results of settling time of FOMNN.

In [65], a new model is designed to counter the fixed-time synchronization problem. The approach takes two general inequalities, extended Cauchy-Schwarz inequality and generalized derivative of the fractional-order absolute value function, to realize the analysis with the acquisition of this less conservative fixed time. Further, two new inequalities to deal with global Mittag-Leffler synchronization are deduced and compared with the existing inequalities in [66]. Also, with Lyapunov theory, two more novel functionals are constructed along with multiple more flexible criteria and a numerical example to demonstrate the proposal’s effect on this problem. Similarly, a benchmark technique is used in [67] for the delayed fractional-order systems of multidimensional-valued memristive neural networks stability. The proposed Lyapunov method with several new lemmas has shown many advantages, including lower conservatism and higher flexibility. Using impulsive and effective analysis techniques, the system’s dynamic behavior is studied in [68]. The Lyapunov method obtains some specific conditions to ensure the stability and passivity of the Memristor-based fractional-order competitive neural networks. To counter stability issues, the technique in [69] uses the framework of Filippov solutions with suitable Lyapunov-functional and differential inclusion theoretical analysis. By employing specific order of value, fractional-order stability of orders 0 to 1 and 1 to
2 are discussed separately by creating a sufficient criterion using Laplace transform and Mittag-Leffler function and Generalized Gronwall inequality. Original fractional-order quaternion-valued systems are divided into four fractional-order real-valued systems based on the non-commutativity of quaternion multiplication. Sufficient conditions are established in [70] by employing Lyapunov fractional-order derivative, fractional-order differential inclusions, global Mittag-Leffler stability, and set-valued maps.

2.6. Fractional-Order Complex Valued Neural Networks

Stability analysis of FOCVNN has been studied over the years. In 2014, a FOCVNN had been proposed using fractional-order differential equations to examine the stability of FOCVNN using memristor-based NNs [71]. In the same year, the work in [72] has presented a complex-valued recurrent NN to find a proper activation function in various complex processes. Continuous FOCVNN with a time delay algorithm has been equipped in the model. In [73], the dissipativity of NNs with time delay is analyzed using a FOCVNN model developed with Lyapunov functions and fractional Halanay inequality and Caputo fractional-order differential equations, which helps in the study of linear, non-linear systems and economic systems.

2.7. Other Neural Networks

2.7.1. Fuzzy-based Neural Networks

In [74], proposed a T-S fuzzy-based NN model using fuzzy Lyapunov synthesis. This paper equipped the Adams-Bashforth-Moulton algorithm and synchronized fractional-order Duffing-Holmes chaotic system. Also, as an advancement in this domain, Recurrent Fuzzy NN is proposed in 2020, which uses sliding mode control and fractional calculus [75]. This model helps in fractional sliding mode control for the micro gyroscope. A fractional-order learning algorithm using Grünwald-Letnikov’s and Riemann-Liouville’s definition is used to develop the model where a two-loop recurrent fuzzy NN is employed to approximate the system uncertainties.

2.7.2. Feedback

In [76], the paper has proposed fractional-order modeling, along with an analysis of stability and control of two robotic manipulators. The pole placement method has been used to derive the control law. Here, the research in the time domain is done using the Mittag-Leffler approach.

2.7.3. Backpropagation

In [77], the screen variables have been screened using the adaptive lasso method, and the various neural network models are established by choosing variables for the seven countries. This paper aims to screen the economic growth in these seven countries, and the local quadratic approximation algorithm has also been implemented.

2.7.4. Forward-Rolling Empirical Decomposition Method

The Forward-Rolling Empirical Decomposition method algorithm uses the fractional calculus predicting model and principal component analysis method to propose a stochastically new financial stock market analysis and financial pricing of stocks model based on Taylor’s formula [78].

2.7.5. NARX Neural Network

In [79], proposed a NARX Neural Network model using fractional-order calculus and a backpropagation algorithm. This model helps in the reduction of large-scale systems with fractional-order non-linear structures. Also, a new type of update rule using fractional calculus is obtained, which has provided good results in fractional-order system model reduction [80].

2.7.6. ResNet

In 2019, introduced a generalization methodology for automatically selecting the activation functions inside NN, which led the neurons within the network to adjust their activation functions to precisely fit the input data and considerably reduce the output error [81].

2.7.7. Multilayer Perceptron Network

By using CNN and PSO, in 2019, Brazilian researchers equipped the fractional-order Darwinian PSO segmentation algorithm K-fold cross-validation technique to propose a method recognizing plotting polynomials using a hybrid segmentation method by fractional calculation [82].

2.7.8. Inverse and Forward Problem

In [83], a variable-order based fractional model is proposed. This model can predict the mean fluid velocity profile, and Reynolds stresses of the fluid with accuracy better than 1% for any range of Reynolds number. Direct numerical simulation data and a Physics-Informed Neural Network have been used to obtain the fractional-order.
Fig. 3. Classification of various fractional-order neural network structures.

The summary of these advances in fractional-order neural networks in terms of structure, learning algorithm, and others is given in Table 1. From the table, the classification of various fractional-order neural network structures is shown in Fig. 3. Further, the different types of stability analysis approaches used in the literature are summarized in Fig. 4.

3. Applications of Fractional-Order Neural Networks

This section presents the critical review on the application of FONNs in various fields such as system identification, control, optimization, stability, and synchronization.

3.1. System Identification and Control

In 2011, a T-S fuzzy-based NN was proposed to synchronize fractional-order Duffing-Holmes chaotic systems [74]. The researchers in 2012 used $\alpha$-exponential stability. They produced a novel fraction order differential inequality that helps in information processing, stimulus anticipation, and frequency independent shifts of oscillatory neural firing [18]. A FOHNN based controller synchronizes a class of Fractional-order Chaotic Neural Networks [37]. In [25], a controller investigating the parameter identification and synchronization problem of FONN with time-delays has a direct behavior of the non-linear dynamical systems. In 2014, the authors studied an Impulsive NN of fractional-order Caputo system of varying time delays and proposed a linear non-impulsive controller [43]. With the help of BAM NN, the Caputo fractional-order BAM neural network is established and analyzed the finite-time stability for pattern recognition and automatic control [54]. In 2016, the neural net’s dissipative analysis was studied, which is used to analyze linear, non-linear, and economic systems [73]. In 2017, parameter estimation of NN models was proposed using adaptive control methods [33]. FONN with multiple time-delays is investigated, and some stability criteria are derived with which it is used to control FONNs with or without sector nonlinearities in control inputs [44]. In 2019, fractional calculation and classification proposed a hybrid segmentation method through CNN to recognize simple handwritten polynomials [78]. In [88], the paper researches the stability and bifurcation of a FONN with double delays in application to system control. Also, in the same year, the authors of [83] have used direct numerical simulations data and designed a physics-informed neural network to obtain an order that helps in a turbulent Couette flow system. In 2020, the indirect method for solving a class of DFOCPs had been based on the NN approach and optimal control problems [13]. Model reduction of large-scale systems with Fractional-order non-linear structure is discussed, and a new type of fractional update rule is obtained [79]. A simple and more accurate FONN model for system identification with minimal parameters is received in the paper [75].

3.2. Stability Analysis & Synchronization

In [1], the paper analyzes FONNs with double delays for their stability and bifurcation. The bifurcation results of the FONNs have been deduced once the leakage delays are not identical to the communication delays. Further research has been conducted on a FONN with two unequal delays for its stability and bifurcation [24]. It has been obtained by intercalating one delay and taking the remnant delay as a bifurcation parameter. This paper aimed to analyze the NNs with different delays for their bifurcation problem and generalize the derived results to more high-order FONNs with numerous delays. The authors of [23] examine the stability of both FONNs with and without delay using S-procedure. The analysis of quaternion-valued FONNs with leakage and discrete delays for their synchronization is reported in [59]. The investigation has been carried out by applying the
<table>
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<th>Year</th>
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properties of quaternion and Caputo derivatives. Further, the global asymptotical stability condition, comparison theory, and global synchronization for FONNs with multiple time delays have been reported [89]. Here, measures like memristor-based anti-synchronization control have been applied. In [22], sufficient conditions have been obtained on the asymptotic stability for Riemann-Liouville FONNs with discrete and distributed delays. Here, the Lipschitz continuous functions have been used as activation functions. Similar analysis on the NNs with time delays for their dissipativity has been reported in [73]. Here, the fractional Halanay inequality has been used to obtain new algebraic conditions. This new Lyapunov function guarantees the global asymptotic stability of FVCNNs with time delays.

In [40], the paper investigates the global Mittag-Leffler stability for FOHNN with impulsive effects. Here, the \(tanh\) function has been used as the activation function. The global synchronization for FOMNN with parameter uncertainty has been obtained in [55]. In [29], the paper aimed to establish a framework for the fundamental stability analysis of the FONNs. Here, the authors worked towards relaxing the knowledge requirements of system uncertainties in the controller design. For activation functions, here, a \(tanh\)-non-linear function has been used.

On the other hand, the work reported in [90] compares and studies the linear delayed systems' stability. Lipschitz continuous functions have been used here as activation functions. The work in [21] discusses the class of delayed FONNs finite-time stability. Here, two effective criteria were derived to ensure the finite-time stability of fractional-order systems. For activation function, \(tanh\) has been used here. The uniform stability analysis conditions for FOCNNs with time delays have been reported in [72]. The paper [71] investigates the stability of memristor-based FOCVNN with time delays. CNN's have complex-valued states, connection weights, and activation functions. In [53], the paper studies the dynamic behavior of the class of FOMNN.

The theoretical stability analysis for FOHNNs with time delays has been reported in [38] for two FOHNNs with ring structures with time delays. Also, the work derived the corresponding sufficient conditions for the system's stability. The stability analysis of FOBAM with delays is reported in [52]. The investigation is based on the inequality technique. Also, Lipschitz continuous functions have been used as activation functions. In [19], the paper presents a sufficient condition for a class of FONNs with time delay has been given to ensure the networks' uniform stability. Here, piecewise linear and Lipschitz continuous functions have been used as activation functions. The finite-time stability for a class of FONNs with time delays has been reported in [20]. The analysis helps in the design and application of fractional networks. The work in [37] synchronizes a class of chaotic FONNs by using Mittag-Leffler functions. Here, Lipschitz continuous function has been used as an activation function. The theoretical stability analyses of FOHNN in [36] reported that the FONNs exhibit chaotic dynamic behavior. A unified form is first established in [92] for fractional-order multidimensional-valued BAM neural networks. A criterion is derived from new LKFs in a vector form and two new inequalities. This has shown higher flexibility, more negligible computation, more diversity, and lower conservatism. The work presented in [93] proposed a proper LKF by using the differential inclusions to achieve global asymptotic stability of fractional-order uncertain BAM competitive neural networks. This is based on the LMI technique and has shown the feasibility and effectiveness of this method.

A novel fractional-order differential inequality has been introduced, i.e., the \(\alpha\)-exponential stability in [18]. Also, some effective criteria have been derived for such kind of stability. Here, Lipschitz and \(tanh\) functions have been used for activation functions. A novel approach using fuzzy and sliding models has been proposed to synchronize the fractional-order Duffing-Holmes chaotic system [74]. The theoretical stability analysis for the FOHNN has been presented in [35]. The proposed study is the first attempt towards theoretically characterizing the Hopfield bifurcation phenomena in FONN. The \(tanh\) function has been used here as the activation function.

### 3.3. Optimization Problems

In 2011, an attempt to solve Fractional-order differential equations using NN was made using the swarm intelligence technique and log sigmoid function as activation function. Statistical data such as fitness values, best, worst, standard deviation are used as performance measures [11]. In 2016 Shuo Zhang et al. analyzed the dynamic properties of the FOMNN system's global attractivity. Firstly, a growth condition is presented. Later, the global Mittag-Leffler stability for a FOMNN system is analyzed. In 2020, solving the Fractional-order partial differential equations using NN is proposed. The analysis has been applied in various problems such as fractional wave equations, boundary value problems, fractional heat conduction equations [12]. In the same year, an approach using NN in solving the delay fractional optimal control problems by an indirect method is presented. Suitable transfer functions are used as activation functions [13, 94, 95, 96, 97]. Within the year 2021, a way to numerically approximate the initial and boundary value problems by using the Haar wavelets technique has been proposed. The proposed method is applied to linear and non-linear fractional-order differential equations, and numerical results show that the proposed method is efficient [84]. Later, an efficient fractional global learning machine called Fragmachine was developed to find a model's global optimal solution and determine the optimal search path [16]. Also, a quantum backpropagation algorithm based on
the fractional Grünwald–Letnikov is proposed in [15] to reduce the convergence speed and error.

3.4. Communication Systems

In 2012, the researchers investigated the hyperchaotic phenomena in the newly proposed FOCNN model in the application of secure communication to improve the system’s security [42]. In 2013 Xia Huang et al. investigated complex dynamics of FOHNN with time delays. They discovered a wide variety of interesting dynamic behavior of the system, such as periodic and chaotic [91]. In 2016 the global projective synchronization of non-identical FONN based on sliding mode control technique was studied, and sufficient criteria were derived [27]. In 2017 an attempt to explore the Fractional-order application of a FONN system synchronization in the areas of secure communication was made. Also, they created an integral sliding surface model, and the generalized projective synchronization of the FONN system with time delays is explored [28].

4. Conclusion

The advances made in the field of fractional-order neural networks along with the development of their structures, learning algorithm, methods, dynamics in the last decade are discussed and summarized in this paper. Various forms of fractional-order neural networks such as FFNN, FOHNN, FOCNN, FOMNN, FOCVNN, and other networks are analyzed critically. Also, a review of applications of these neural networks in system identification and control, stability analysis and synchronization, optimization, and communication is presented.

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References


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