Engineering Journal issn: 0125-8281

Article

New Results on Positive Realness in the Presence of Delayed Dynamics

Manuel De la Sen[1,a,*] **and Asier Ibeas**[2,b]

1 Department of Electricity and Electronics, Faculty of Science and Technology, University of the Basque Country, Barrio Sarriena, Leioa, Bizkaia, Spain
2 Department of Telecommunications and Systems Engineering, Universitat Autònoma de Barcelona, Barcelona, Spain
E-mail: [a]manuel.delasen@ehu.eus (Corresponding author) and [b]Asier.Ibeas@uab.cat

Abstract. Positive realness is a very important tool for the achievement of hyperstability and passivity of dynamic systems. This paper is devoted to extend some positive realness results of transfer functions in the presence of point-delayed delayed dynamics. Sufficiency-type conditions which guarantee the positive realness of delayed transfer functions under point delays are given. The value of the direct input-output interconnection gain is seen to be crucial in the performed analysis. The relevance of the results in potential applications rely in the importance of the hyperstability property of closed-loop systems under non-linear and time-varying controller devices. In fact, if the feed-forward controlled plant has a strictly positive real transfer function, then the closed-loop system is asymptotically hyperstable, that is, globally asymptotically Lyapunov's stable for any non-linear time-varying controller which belongs to a hyperstable class defined as that which satisfies a Popov's type inequality.

Keywords: Hyperstability, point internal and external delays, positive realness, transfer function.

ENGINEERING JOURNAL Volume 23 Issue 1
Received 8 June 2018
Accepted 19 October 2018
Published 31 January 2019
Online at http://www.engj.org/
DOI:10.4186/ej.2019.23.1.75

DOI:10.4186/ej.2019.23.1.75

1. Introduction

It is well- known in the background literature the key role of the positive realness properties in certain extended stability results like hyperstability and passivity. Intuitively speaking, positive realness of a dynamic system in the feed-forward loop of a closed-loop configuration of a control system guarantees that its input-output energy is non- negative for all time. This fact allows the closed-loop stabilization of controlled systems under very wide classes of non-linear time-varying controllers satisfying Popov's type time-integral inequalities provided that the feed-forward loops are given by positive real transfer functions. Such a property is usually known as the closed-loop hyperstability property, that is, the global stability of the closed-loop system for all the class of controllers satisfying such Popov's-type inequalities, [1-8]. It is of interest to investigate how such properties are kept in the presence of point delays in the feed-forward loops or, in other words, in the case when the feed-forward transfer function is subject to point delays.

This paper presents some new positive realness results for transfer functions in the presence of dynamics involving point delays. Such delays become of inherent and relevant importance in modeling many real-life processes of biological, medical, motion or diffusion characteristics (see, for instance, [9–12]. A main motivation of the study in inspired in the relevance of stability and absolute stability in the study of dynamic systems used as models in problems of Mechanical Engineering, Electrical Engineering, Electric Circuitry and Automation (see, for instance, [9], [11], [13, 14] and references therein) and the fact that hyperstability is a generalized concept of absolute stability to the case when the nonlinear control device can be, in general, time-varying rather than just static while the dynamics is subject to delays. Other references related to stability, hyperstability and positive realness with some applications can be found in [8, 15–20]. The approach which is used through the paper to deal with the positive realness of time-delay systems relies on the properties of the transfer functions rather than in those of the state-space descriptions as, for instance, in [21-22] and references therein. It is illustrated that positive realness of a delay-free transfer function can be lost in the presence of delays and that the delays are not favoring the achievement of the positive realness of a transfer function related to the delay-free case. These features are examined in Section 2 together with the description of known properties of delay-free transfer functions as well as the statements and proofs of some more related properties concerning the presence of delays. It is seen, in particular, that, if the real positivity in the delay-free case is not strict, then the property is easily lost in the presence of delays. Section 3 gives some results and proofs on positive realness for the case when the transfer function is composed of a parallel time-delay free one with another one which is stable and subject to an external delay. Such a delay is physically associated to be present in either the input or the output of the dynamic system described by the second block of such a composed transfer function as it is common in certain physical systems like war/peace models, sunflower equation, signal transmission etc. . That section also includes other more powerful main results for the case when the transfer function is described by a quotient of quasi-polynomials composed of additive delay-free and delayed terms. That further description is, in particular, useful to describe the case of internal (i.e. in the state) delays. Finally, Section 3 includes also a further result by considering the case of a combined presence of internal and external delays, that is, in the state components and the input and/or output. Some numerical simulations are presented and discussed in Section 4 where some frequency responses and related Nyquist plots are given for transfer functions subject to internal and external point delays. The corresponding time responses under unity step inputs are also given. The applicability to control problems as well as a physical example which is based on a proposed electric circuit subject to a delay are given and discussed in section 6. Finally, conclusions end the paper.

2. Extended Positive Realness Results

Some useful definitions about positive realness to be used later on follow:

Definition 1. A function $G(s)$ of the complex variable $s=\sigma+\boldsymbol{i}\omega$, where $\boldsymbol{i}=\sqrt{-1}$ is the complex imaginary unit, is positive real $(G\in\{PR\})$ if $G(\sigma)$ is real, i.e. if $G(Re\ s)=Re\ G(Re\ s))$ and $Re\,G(s)\geq 0$ for all $Re\ s>0$.

Definition 2. $G\in\{PR\}$ is strictly positive real $(G\in\{SPR\})$ if $G_\varepsilon\in\{PR\}$ for some real $\varepsilon>0$ where $G_\varepsilon(s)=G(s-\varepsilon)$; $\forall s\in\boldsymbol{C}$.

Definition 3. $G \in \{PR\}$ is strictly positive real in the particular case ($G \in \{SPR_0\}$) if it is of the form $G(s) = s^{-1}G'(s)$.

Definition 4. $G \in \{SPR\}$ is strongly strictly positive real $(G \in \{SSPR\})$ if $\lim_{Re\, s \to +\infty} Re\, G(s) > 0$.

Observation 1: Note that

$G \in \{SSPR\} \Rightarrow G \in \{SPR\} \Rightarrow G \in \{PR\}$ but the reverse implications are not true, in general.

The following results emphasizes two facts, namely, a) the positive realness of a free-delay transfer function can be lost in the presence of delays; b) the positive realness of a delayed transfer function is not necessarily achieved with the incorporation of delays.

Proposition 1. Assume that $G \in \{PR\}$ and $\Delta_G \neq \varnothing$ where
$\Delta_G = \{\omega \in \mathbf{R}_{0+} : Im\, \mathrm{G}(\boldsymbol{i}\omega) < -|Re\, \mathrm{G}(\boldsymbol{i}\omega)|\, ctg(h\omega)\}$. Then $G(s)e^{-hs} \notin \{PR\}$.

Proof: Note that
$Re\, G(\boldsymbol{i}\,\omega)e^{-\boldsymbol{i}\,h\omega} = Re\, G(\boldsymbol{i}\,\omega)cos\,(h\omega) + Im\, G(\boldsymbol{i}\,\omega)\, sin(h\omega) = |Re\, G(\boldsymbol{i}\,\omega)|\, cos\,(h\omega) + Im\, G(\boldsymbol{i}\,\omega)\, sin(h\omega) < 0$
for some $\omega \in \mathbf{R}_{0+}$ if $Im\, G(\boldsymbol{i}\,\omega) < -|Re\, G(\boldsymbol{i}\,\omega)|\, ctg\,(h\omega)$. □

Proposition 2. Assume that $G \notin \{PR\}$ and $\Delta'_G \neq \varnothing$ where $\Delta'_G = \{\omega \in \mathbf{R}_{0+} : Im\, \mathrm{G}(\boldsymbol{i}\omega) < |Re\, \mathrm{G}(\boldsymbol{i}\omega)|\, ctg h\omega\}$. Then $G(s)e^{-hs} \notin \{PR\}$.

Proof: Note that, since $G \notin \{PR\}$, there is $\omega \in \mathbf{R}_{0+}$ such that
$Re\, G(\boldsymbol{i}\,\omega)e^{-\boldsymbol{i}\,h\omega} = -|Re\, G(\boldsymbol{i}\,\omega)|cos\,(h\omega) + Im\, G(\boldsymbol{i}\,\omega)\, sin(h\omega) < 0$
for some $\omega \in \mathbf{R}_{0+}$ if $Im\, G(\boldsymbol{i}\,\omega) < |Re\, G(\boldsymbol{i}\,\omega)|\, ctg\,(h\omega)$. □

Example 1. Take $G(s) = 1/s$ which is positive real. Then, $Re\left(\frac{e^{-\boldsymbol{i}h\omega}}{\boldsymbol{i}\omega}\right) = -\frac{sin(h\omega)}{\omega} < 0$ if $\omega \in \left(\frac{0}{h}, \frac{\pi}{h}\right)$.Thus, $G(s)e^{-hs} \notin \{PR\}$ for any given delay $h > 0$.

Example 2. Take $G(s) = -1/s$ which is not positive real. Then, $Re\left(-\frac{e^{-\boldsymbol{i}h\omega}}{\boldsymbol{i}\omega}\right) = \frac{sin(h\omega)}{\omega} < 0$ if $\omega \in \left(\frac{\pi}{h}, \frac{2\pi}{h}\right)$. Thus, $G(s)e^{-hs} \notin \{PR\}$ for any given delay $h > 0$.

Example 3. Take $G(s) = \frac{s+a}{s+b}$ which is strongly strictly positive real if $min(a, b) > 0$. Then, $Re\left(G(\boldsymbol{i}\omega)e^{-\boldsymbol{i}h\omega}\right) = \frac{(ab + \omega^2)cos\,(h\omega) + (b-a)\omega\, sin(h\omega)}{a^2 + b^2} < 0$ if $\omega = \frac{2n\pi}{h}$ for any positive integer n. Thus, $G(s)e^{-hs} \notin \{PR\}$ for any given delay $h > 0$.

Properties of positive real functions.

1. If $G \in \{PR\}$ then it is analytic in the open right half-plane.
2. If $G(s)$ is rational then $G(Re\, s) = Re\, G(Re\, s)$ if the coefficients of its numerator and denominator polynomials are all real.
3. If $G \in \{PR\}$ with poles on the imaginary axis then they must be distinct and have positive residues.
4. If $G \in \{SPR\}$ then it is analytic in the closed right-half-plane (except for the case when $G \in \{SPR_0\}$), that it, $G \in \boldsymbol{H}_\infty$ so that if it is rational and proper then $G \in \boldsymbol{RH}_\infty$.

Observation 2: It is said that the transfer function $G(s)$ is in the simplest particular case form following Popov's classical terminology since it has a simple pole at the origin.

5. If $G \in \{PR\}$ then $G^{-1} \in \{PR\}$.
6. If $G \in \{PR\}$ then its relative degree, or order, (i.e. its pole-zero excess) is +1, -1 or 0.
7. If $G \in \{PR\}$ then $Re\, G(\boldsymbol{i}\omega) \geq 0$; $\forall \omega \in \boldsymbol{R}$, if $G \in \{SPR\}$ then $Re\, G(\boldsymbol{i}\omega) > 0$; $\forall \omega \in \boldsymbol{R}$ and if $G \in \{SSPR\}$ hen $Re\, G(\boldsymbol{i}\omega) > 0$; $\forall \omega \in cl\boldsymbol{R}$. Because of the real axis symmetry property of the hodographs $G(\boldsymbol{i}\omega)$ for $\omega \in cl\boldsymbol{R}$ those properties are proved by just testing for all $\omega \in \boldsymbol{R}_{0+} \equiv \boldsymbol{R}_{+} \cup \{0\}$, where $cl\boldsymbol{R} = \boldsymbol{R} \cup \{\pm\infty\}$ is the extended real line, i.e. the closure of the real set.
8. If G is rational and proper with real numerator and denominator coefficients then: a) $G \in \{PR\}$ if and only if it is either stable or critically stable with single poles all with positive residuals, and $Re\, G(\boldsymbol{i}\omega) \geq 0$; $\forall \omega \in \boldsymbol{R}_{0+}$; b) $G \in \{SPR\}$ if and only if it is in $\boldsymbol{RH}_{\infty}$, $Re\, G(\boldsymbol{i}\omega) > 0$; $\forall \omega \in \boldsymbol{R}_{0+}$ and $\lim_{\omega \to \infty} \omega^2\, Re\, G(\boldsymbol{i}\omega) > 0$; c) $G \in \{SSPR\}$ if and only if it is in $\boldsymbol{RH}_{\infty}$ and $Re\, G(\boldsymbol{i}\omega) > 0$; $\forall \omega \in \boldsymbol{R}_{0+}$ and $\lim_{\omega \to +\infty} Re\, G(\boldsymbol{i}\omega) > 0$.
9. If a proper complex function $G \in \{SSPR\}$ then it has necessarily a zero relative degree.
10. A rational complex function $G \in \{SPR_0\}$ if and only if $Im\, G'(s-\varepsilon) > 0$ for all $Re\, s \geq 0$ and some real $\varepsilon > 0$, where $G'(s) = sG(s)$.
11. If $G \in \{SPR_0\}$ and proper then $G'(s)$ is of relative order of at most $+1$ and of strictly Hurwitz numerator and denominator polynomials (then $G'(s)$ being biproper, i.e. proper with proper inverse so that with zero relative degree, and in $\boldsymbol{RH}_{\infty}$) with
$Re\left(\boldsymbol{i}G'^{*}(\boldsymbol{i}\omega)\right) = Im\, G'(\boldsymbol{i}\omega) = -Re\left(\boldsymbol{i}G'(\boldsymbol{i}\omega)\right) = -Re\left(\boldsymbol{i}G'(-\boldsymbol{i}\omega)\right) > 0$; $\forall \omega \in cl\boldsymbol{R}_{0+}$
where the superscript * denotes the complex conjugate; or $G'(s)$ has relative order +1 with Hurwitz numerator and Hurwitz denominator if $G(s)$ is biproper (since then $G'(s)$ is not proper and has a zero-pole cancellation at s=0).

Proof of Properties 10-11

Direct calculations yield:

$$Re\, G(\sigma + \boldsymbol{i}\omega) = \frac{\sigma\, Re\, G'(\sigma + \boldsymbol{i}\omega)}{\sigma^2 + \omega^2} + \frac{\omega\, Im\, G'(\sigma + \boldsymbol{i}\omega)}{\sigma^2 + \omega^2}; \ \forall \sigma, \omega \in \boldsymbol{R}$$

Then, if $Im\, G'(\boldsymbol{i}\omega) > 0$; $\forall \omega \in \boldsymbol{R}_{0+}$:

$$Re\, G(\boldsymbol{i}\omega) = \frac{Im\, G'(\boldsymbol{i}\omega)}{\omega} > 0; \ \forall \omega \in \boldsymbol{R}_{0+}$$

Note that there is a cancellation at $\omega = 0$ in the above formula, since $G \in \{SPR_0\}$, so that $Re\, G(\boldsymbol{i}0) > 0$, since, and also

$$\omega^2\, Re\, G(\boldsymbol{i}\omega) = \omega\, Im\, G'(\boldsymbol{i}\omega) > 0; \ \forall \omega \in cl\boldsymbol{R}_{+}$$

so that $G' \in \{SPR\}$ since $Re\, G'(\boldsymbol{i}\omega) = -\omega\, Im(G(\boldsymbol{i}\omega)) = |\omega\, Im(G(\boldsymbol{i}\omega))|$; $\forall \omega \in cl\boldsymbol{R}_{+}$ and $Re\, G'(0) = -0 \times Im(G(0)) = |0\, Im(G(0))| > 0$ since $Im(G(0)) = \infty$ since $G \in \{SPR_0\}$. Then $Im\, G'(\boldsymbol{i}\omega) > 0$; $\forall \omega \in cl\boldsymbol{R}_{0+}$ and, because of the symmetry of the frequency response hodograph, $Im\, G'(\boldsymbol{i}\omega) > 0\ \forall \omega \in cl\boldsymbol{R}_{+}$. Property 11 is proved and Property 10 is equivalent. □

Properties 7-11 are used for the proofs of most of the subsequent results on positive realness.

Define the following amounts for subsequent developments:

$$d_0 = min\left\{Re\ G_0(s):\ Re\ s \geq 0\right\}$$
$$d_{dm} = min\left\{Re\ G_d(s):\ Re\ s \geq 0\right\}$$
$$d_{dM} = max\left\{Re\ G_d(s):\ Re\ s \geq 0\right\}$$

Note that such numbers are the input-output interconnection gains of the respective transfer functions . In the event any of them is nonzero the corresponding transfer function is biproper, that is, it has the same number of poles and zeros so that its inverse is also physically realizable.

The following results hold:

Assertion 1. Assume that $G(s) = G_0(s) + G_d(s)$ and that $G_0 \in \{SSPR\}$. If $d_{dn} \geq 0$ then $G \in \{SSPR\}$.

Proof: Since $G_0 \in \{SSPR\}$, $d_0 > 0$ then $G_0 \in \boldsymbol{RH}_\infty$ is of zero relative degree, thus proper but not strictly proper. Also, since $d_{dn} \geq 0$ then $G_d \in \{PR\}$. Then, $\underset{Re\ s \geq 0}{min}\ Re\, G(s) = d_0 + d_{dn} \geq d_0 > 0$. □

Assertion 2. Assume that $G(s) = G_0(s) + G_d(s)$ and that $G_0 \in \{SSPR\}$ and $d_{dn} < 0$. Then $G \in \{SSPR\}$ if and only if $d_0 > |d_{dn}|$.

Proof: $\underset{Re\ s \geq 0}{min}\ Re\, G(s) = d_0 - |d_{dn}| > 0$ and sufficiency follows.

The necessity follows since the contrary constraint $d_0 \leq |d_{dn}|$ leads to the contradiction $0 = |d_{dn}| - |d_{dn}| \geq d_0 - |d_{dn}| > 0$. □

3. Extended Positive Realness Results in the Presence of Point Delays

The subsequent result concerns a transfer function subject to a delay- free component together with a parallel point delayed contribution.

Assertion 3. Assume that $G_h(s) = G_0(s) + e^{-hs} G_d(s)$ for any given delay $h \geq 0$ and that $d_{dn} \geq 1$. Then, the following properties hold:

(i) If $G_0 \in \{SSPR\}$ then $G_h \in \{SSPR\}$.
(ii) If $G_0 \in \{SPR\}$ then $G_h \in \{SPR\}$.
(iii) If $G_0 \in \{PR\}$ then $G_h \in \{PR\}$.

Proof: Since $G_0 \in \{SSPR\}$ then $d_0 > 0$ with $G_0 \in \boldsymbol{RH}_\infty$ being of zero relative degree. Also, since $d_{dn} \geq 1$ then $G_d \in \{SSPR\}$ and $e^{-hs} G_d(s) \in \boldsymbol{RH}_\infty$ with $\underset{Re\ s \geq 0}{min}\ Re\ e^{-hs} G_d(s) \geq 0$ so that $e^{-hs} G_d(s) \in \{PR\}$. Thus, $\underset{Re\ s \geq 0}{min}\ Re\, G_h(s) = d_0 > 0$ and $G_h \in \{SSPR\}$ leading to Property (i). The proofs of Properties [(i)-(ii)] follow in the same way. □

Assertion 4. Assume that $G_h(s) = G_0(s) + e^{-hs} G_d(s)$ for any given $h \geq 0$ and that $G_d(s) \in \boldsymbol{RH}_\infty$ with $\|G_d\|_\infty \leq d_{dM}$. Then, the following properties hold:

(i) If $G_0 \in \{SSPR\}$ then $G_h \in \{SSPR\}$ if $d_0 > d_{dM}$.
(ii) If $G_0 \in \{SPR\}$ then $G_h \in \{SPR\}$ if $d_0 \geq d_{dM}$.

(iii) Assume that $G_d(s)=f(s)G_0(s)$. Then, Property (i) (respectively, Property (ii)) holds if $f(s)$ is rational with strictly stable poles or constant satisfying $\sup_{Re\, s\geq 0} f(s)=f_\infty \leq \frac{d_{dM}}{d_0+\varepsilon_\infty}$ with $d_0>d_{dM}$ (respectively, with $d_0 \geq d_{dM}$) if $\varepsilon_\infty = \left\| G'_0(s)\right\|_\infty = \sup_{\omega\in \boldsymbol{R}_{0+}} \left| G'_0(\boldsymbol{i}\omega)\right|$ ($\boldsymbol{i}=\sqrt{-1}$) ,where $G'_0(s)=(G_0(s)-d_0)$ is strictly proper.

Proof: Since $G_0 \in \{SSPR\}$ then $d_0>0$ with $G_0 \in \boldsymbol{RH}_\infty$ being of a zero relative degree. Also, $\sup_{Re\, s\geq 0}\left| Re\, e^{-hs}G_d(s)\right| \leq \left\| e^{-hs}G_d(s)\right\|_\infty = \sup_{\omega\in \boldsymbol{R}_{0+}} \left| G_d(\boldsymbol{i}\omega)\right| \leq d_{dM}$. Then, $\min_{Re\, s\geq 0} Re\, G_h(s)\geq d_0-d_{dM}>0$ and $G_h \in \{SSPR\}$. Property (i) has been proved. The proof of Property (ii) is close with $\min_{Re\, s\geq 0} Re\, G_h(s)\geq d_0-d_{dM}\geq 0$ leading to $G_h \in \{SPR\}$. Property (iii) is proved directly from Properties (i)-(ii) by taking into account that:

$$\left\| e^{-hs}G_d(s)\right\|_\infty = \left\| e^{-hs} f(s)\left(G'_0(s)+d_0\right)\right\|_\infty \leq f_\infty(\varepsilon_\infty + d_0)\leq d_{dM}\,.$$

□

The following result is concerned with positive realness conditions independent of the delay size for a special transfer function which has the arbitrary point delay h acting jointly as an internal delay on the state and an external one on the output.

Theorem 1. Consider a delay-dependent state-space realizable transfer function of the form:

$$G_h(s)=\frac{N_0(s)+e^{-hs}N_d(s)}{D_0(s)+e^{-hs}D_d(s)} \tag{1}$$

where $N_0(s)$, $D_0(s)$, $N_d(s)$, $D_d(s)$ are polynomials such that $d_0=\min_{\omega\in \boldsymbol{R}_{0+}} Re\, G_0(\boldsymbol{i}\,\omega)$, $G_0(s)=N_0(s)/D_0(s)$ and $G_d(s)=N_d(s)/D_d(s)$ are proper, and

$$d_0 > \sup_{\omega\in \boldsymbol{R}_{0+}} \left| \frac{N_d(\boldsymbol{i}\,\omega)-D_d(\boldsymbol{i}\,\omega)G_0(\boldsymbol{i}\,\omega)}{D_0(\boldsymbol{i}\,\omega)+e^{-\boldsymbol{i}\,h\omega}D_d(\boldsymbol{i}\,\omega)}\right| \tag{2}$$

The following properties hold:

(i) If $D_d(s)/D_0(s)$ is strictly bounded real (i.e. it is in $\boldsymbol{RH}_\infty$ with real coefficients and with H_∞ norm being strictly less than unity) then $p\left(s,e^{-hs}\right)$ is Hurwitz irrespective of the delay size and $G_h \in \{SSPR\}$ for the given delay h .

(ii) If $D_0(s)/D_d(s)$ is strictly bounded real (i.e. it is in $\boldsymbol{RH}_\infty$ with real coefficients and with H_∞ norm being strictly less than unity) then $p\left(s,e^{-hs}\right)$ is Hurwitz irrespective of the delay size and $G_h \in \{SSPR\}$ for the given delay h .

(iii) $G_h \in \{SSPR\}$ is guaranteed in Properties [(i) -(ii)] irrespective of the delay size if

$$d_0 > \sup_{\omega\in \boldsymbol{R}_{0+}} \frac{\left| N_d(\boldsymbol{i}\,\omega)-D_d(\boldsymbol{i}\,\omega)G_0(\boldsymbol{i}\,\omega)\right|}{\left| \left| D_0(\boldsymbol{i}\,\omega)\right| - \left| D_d(\boldsymbol{i}\,\omega)\right| \right|} \tag{3}$$

Proof: Note that $G_h(s)$ can be rewritten by separating the delay- free $G_0(s)$ part and delay-dependent part as follows:

$$
\begin{aligned}
G_h(s) &= \frac{N_0(s)}{D_0(s)} + \frac{e^{-hs}\left(N_d(s) - D_d(s)N_0(s)/D_0(s)\right)}{D_0(s) + e^{-hs}D_d(s)} \\
&= G_0(s) + \frac{e^{-hs}\left(N_d(s) - D_d(s)G_0(s)\right)}{D_0(s) + e^{-hs}D_d(s)} \\
&= G_0(s)\left(1 - \frac{e^{-hs}D_d(s)}{D_0(s) + e^{-hs}D_d(s)}\right) + \frac{e^{-hs}N_d(s)}{D_0(s) + e^{-hs}D_d(s)} \\
&= G_0(s) + \frac{e^{-hs}N_d(s)}{D_0(s) + e^{-hs}D_d(s)} - \frac{e^{-hs}D_d(s)}{D_0(s) + e^{-hs}D_d(s)}G_0(s)
\end{aligned} \tag{4}
$$

$$
Re\left(G_h(s) - G_0(s)\right) = Re\left(\frac{e^{-hs}N_d(s)}{D_0(s) + e^{-hs}D_d(s)} - \frac{e^{-hs}D_d(s)}{D_0(s) + e^{-hs}D_d(s)}G_0(s)\right) \tag{5}
$$

Then,

$$
\begin{aligned}
&\left\| \frac{e^{-hs}N_d(s)}{D_0(s) + e^{-hs}D_d(s)} - \frac{e^{-hs}D_d(s)}{D_0(s) + e^{-hs}D_d(s)}G_0(s) \right\|_\infty \\
&\geq Re\left(G_h(s) - G_0(s)\right) \\
&\qquad \geq -\left\| \frac{e^{-hs}N_d(s)}{D_0(s) + e^{-hs}D_d(s)} - \frac{e^{-hs}D_d(s)}{D_0(s) + e^{-hs}D_d(s)}G_0(s) \right\|_\infty
\end{aligned} \tag{6}
$$

which implies that $Re\,G_h(s) > 0$ for $Re\ s \geq 0$ if

$$
\min_{Re\ s \geq 0} Re\,G_0(s) = d_0 > \sup_{\omega \in \mathbf{R}_{0+}} \left| \frac{N_d(\boldsymbol{i}\omega) - D_d(\boldsymbol{i}\omega)G_0(\boldsymbol{i}\omega)}{D_0(\boldsymbol{i}\omega) + e^{-\boldsymbol{i}h\omega}D_d(\boldsymbol{i}\omega)} \right| \tag{7}
$$

since then

$$
Re\,G_h(s) \geq Re\,G_0(s) - \left\| \frac{N_d(s) - D_d(s)G_0(s)}{D_0(s) + e^{-hs}D_d(s)} \right\|_\infty > 0 \tag{8}
$$

for $Re\ s \geq 0$, where the above H_∞- norm exists if $p\left(s, e^{-hs}\right) = D_0(s) + e^{-hs}D_d(s)$ is a Hurwitz quasi-polynomial. Sufficient conditions for that are proved in Lemma A.1 below. Note that

$$
\begin{aligned}
p\left(s, e^{-hs}\right) &= D_0(s)\left(1 + e^{-hs}D_d(s)/D_0(s)\right) \\
&= D_d(s)e^{-hs}\left(1 + e^{hs}D_0(s)/D_d(s)\right)
\end{aligned} \tag{9}
$$

Lemma A.1 (Auxiliary lemma in the proof of Theorem 1). The following properties hold:

(i) If $D_0(s)$ is Hurwitz and $D_d(s)/D_0(s)$ is proper and strictly bounded real then $p\left(s, e^{-hs}\right)$ is Hurwitz.

(ii) If $D_d(s)$ is Hurwitz and $D_0(s)/D_d(s)$ is proper and strictly bounded real then $p\left(s, e^{-hs}\right)$ is Hurwitz.

Proof: Note that $p\left(s, e^{-hs}\right)-D_0(s)=e^{-hs}D_d(s)$ so that, if $\left|p\left(s, e^{-hs}\right)-D_0(s)\right|=\left|e^{-hs}D_d(s)\right|<\left|D_0(s)\right|$ for $s=\boldsymbol{i}\omega$ and all $\omega\in\boldsymbol{R}_{0+}$, equivalently, if $\left\|\frac{D_0(s)}{D_d(s)}\right\|_\infty = \sup_{\omega\in\boldsymbol{R}_{0+}}\left|\frac{D_0(\boldsymbol{i}\omega)}{D_d(\boldsymbol{i}\omega)}\right|<1$ then $p\left(s, e^{-hs}\right)$ and $D_0(s)$ have the same number of zeros (i.e. none) in $Re\, s\geq 0$ from the Rouché theorem on zeros within the region whose frontier is the closed contour, a Jordan curve, defining the closed right- half-place. As a result, $p\left(s, e^{-hs}\right)$ is Hurwitz since $D_0(s)$ is Hurwitz and Property (i) is proved. Property (ii) is proved in the same way via the alternative identity

$$p\left(s, e^{-hs}\right)-D_d(s)e^{-hs}=D_0(s).$$

End of proof of Lemma A.1. □

Note that, since $d_0>0$ necessarily under the constraint $d_0>\sup_{\omega\in\boldsymbol{R}_{0+}}\left|\frac{N_d(\boldsymbol{i}\omega)-D_d(\boldsymbol{i}\omega)G_0(\boldsymbol{i}\omega)}{D_0(\boldsymbol{i}\omega)+e^{-\boldsymbol{i}h\omega}D_d(\boldsymbol{i}\omega)}\right|$, then $d_0>0$ and $G_0(s)$ is strictly proper. Since $N_d(s)/D_d(s)$ is proper by hypothesis then $G_h(s)$ is strictly proper. Therefore, since $d_0>0$ then $G_0(s)$ and $G_h(s)$ is in $\boldsymbol{RH}_\infty\cap\{SSPR\}$ since $D_0(s)$ and $p\left(s, e^{-hs}\right)$ are Hurwitz from Lemma A.1 (i) , $D_d(s)/D_0(s)$ is proper and strictly bounded real, and

$$Re\,G_h(s)\geq Re\,G_0(s)-\left\|\frac{N_d(s)-D_d(s)G_0(s)}{D_0(s)+e^{-hs}D_d(s)}\right\|_\infty>0\,.$$

Property (i) has been proved. Property (ii) follows in a similar way by using Lemma A.1 (ii). Property (iii) follows since

$$\left(d_0>\sup_{\omega\in\boldsymbol{R}_{0+}}\left|\frac{N_d(\boldsymbol{i}\omega)-D_d(\boldsymbol{i}\omega)G_0(\boldsymbol{i}\omega)}{\left|D_0(\boldsymbol{i}\omega)\right|-\left|D_d(\boldsymbol{i}\omega)\right|}\right|\right)$$

(which is a condition independent on the value of h)

$$\Rightarrow\left(d_0>\sup_{\omega\in\boldsymbol{R}_{0+}}\left|\frac{N_d(\boldsymbol{i}\omega)-D_d(\boldsymbol{i}\omega)G_0(\boldsymbol{i}\omega)}{D_0(\boldsymbol{i}\omega)+e^{-\boldsymbol{i}h\omega}D_d(\boldsymbol{i}\omega)}\right|\right);\forall h\in\boldsymbol{R}_{0+}\,.$$

□

The ordinary differential equation associated with the transfer function $G_h(s)$ under a piecewise-continuous forcing function $u:\boldsymbol{R}_{0+}\to\boldsymbol{R}$ is:

$$D_0(D)y(t)+D_d(D)y(t-h)=N_0(D)u(t)+N_d(D)u(t-h) \tag{10}$$

subject to any given initial conditions where the Laplace operator s has been formally replaced with the time-derivative operator $D=d/dt$. If $D_0(s)=s^n+\overline{D}_0(s)$ is a monic polynomial of degree n being non less than that of $D_d(s)$ then the above differential equation becomes:

$$y^{(n)}(t)+D_d(D)y(t-h)=-\overline{D}_0(D)y(t)+N_0(D)u(t)+N_d(D)u(t-h) \tag{11}$$

for any initial conditions $y^{(i)}(0)=y_{i0}$ for $i=0,1,\ldots,n-1$. Now assume that there are two delays involved in the transfer function which takes the form:

$$G_{hh'}(s)=\frac{N_0(s)+e^{-hs}N_d(s)}{D_0(s)+e^{-h's}D_d(s)} \quad (12)$$

with $h' \neq h$ so that its associated differential equations becomes:

$$\begin{aligned} & y^{(n)}(t)+D_d(D)y(t-h') \\ & =-\overline{D}_0(D)y(t)+N_0(D)u(t)+N_d(D)u(t-h) \end{aligned} \quad (13)$$

for any given initial conditions for any initial conditions $y^{(i)}(0)=y_{i0}$; $i=0,1,\ldots,n-1$. Now, note that

$$G_{hh'}(s)=\frac{N_0(s)}{D_0(s)}+\frac{e^{-hs}\left(N_d(s)-e^{-(h'-h)s}D_d(s)N_0(s)/D_0(s)\right)}{D_0(s)+e^{-h's}D_d(s)}$$

$$\begin{aligned} & =G_0(s)+\frac{e^{-hs}\left(N_d(s)-e^{-(h'-h)s}D_d(s)G_0(s)\right)}{D_0(s)+e^{-h's}D_d(s)} \\ & =G_0(s)\left(1-\frac{e^{-h's}D_d(s)}{D_0(s)+e^{-h's}D_d(s)}\right)+\frac{e^{-hs}N_d(s)}{D_0(s)+e^{-h's}D_d(s)} \\ & =G_0(s)+\frac{e^{-hs}N_d(s)}{D_0(s)+e^{-h's}D_d(s)}-\frac{e^{-h's}D_d(s)}{D_0(s)+e^{-h's}D_d(s)}G_0(s) \end{aligned} \quad (14)$$

Then,

$$\begin{aligned} & Re\left(G_h(s)-G_0(s)\right) \\ & =Re\left(\frac{e^{-hs}N_d(s)}{D_0(s)+e^{-h's}D_d(s)}-\frac{e^{-h's}D_d(s)}{D_0(s)+e^{-h's}D_d(s)}G_0(s)\right) \end{aligned} \quad 15)$$

Then, Theorem 1 becomes extended as follows for the case of the two distinct delays h and h' under a similar proof:

Theorem 2. Consider a delay-dependent state-space realizable transfer function of the form:

$$G_{hh'}(s)=\frac{N_0(s)+e^{-hs}N_d(s)}{D_0(s)+e^{-h's}D_d(s)} \quad (16)$$

with $h' \neq h$ where $N_0(s)$, $D_0(s)$, $N_d(s)$, $D_d(s)$ are polynomials such that $d_0=\min_{\omega \in \boldsymbol{R}_{0+}} Re G_0(\boldsymbol{i}\omega)$, $G_0(s)=N_0(s)/D_0(s)$ and $G_d(s)=N_d(s)/D_d(s)$ are proper, and

$$d_0>\sup_{\omega \in \boldsymbol{R}_{0+}}\left|\frac{N_d(\boldsymbol{i}\omega)-e^{-(h'-h)\boldsymbol{i}\omega}D_d(\boldsymbol{i}\omega)G_0(\boldsymbol{i}\omega)}{D_0(\boldsymbol{i}\omega)+e^{-\boldsymbol{i}h'\omega}D_d(\boldsymbol{i}\omega)}\right| \quad (17)$$

The following properties hold:

(i) If $D_d(s)/D_0(s)$ is strictly bounded real (i.e. it is in $\boldsymbol{RH}_\infty$ with real coefficients and with H_∞ norm being strictly less than unity) then $p\left(s, e^{-h's}\right)$ is Hurwitz independent of h and irrespective of the size of h' and $G_{hh} \in \{SSPR\}$ for the given delays $h \geq 0$ and $h' \geq 0$.

(ii) If $D_0(s)/D_d(s)$ is strictly bounded real (i.e. it is in $\boldsymbol{RH}_\infty$ with real coefficients and with H_∞ norm being strictly less than unity) then $p\left(s, e^{-h's}\right)$ is Hurwitz independent of h and irrespective of the size of h' and $G_{hh'} \in \{SSPR\}$ for the given delays $h \geq 0$ and $h' \geq 0$.

(iii) $G_{hh'} \in \{SSPR\}$ is guaranteed in Properties [(i) -(ii)] irrespective of the delay size provided that

$$d_0 > \sup_{\omega \in \boldsymbol{R}_{0+}} \frac{\left|N_d(\boldsymbol{i}\omega)\right| + \left|D_d(\boldsymbol{i}\omega)G_0(\boldsymbol{i}\omega)\right|}{\left|\left|D_0(\boldsymbol{i}\omega)\right| - \left|D_d(\boldsymbol{i}\omega)\right|\right|}. \qquad \square$$

Now, consider a transfer function in the simplest particular case $G_{hh'}(s) = G'_{hh'}(s)/s$ with $h \neq h'$, where $G'_{hh'}(s) = \frac{N_0(s) + e^{-hs}N_d(s)}{D_0(s) + e^{-h's}D_d(s)}$ with $h' \neq h$ so that its associated differential equations becomes:

$$y^{(n+1)}(t) + D_d(D)y(t-h') = -\overline{D}_0(D)y(t) + N_0(D)u(t) + N_d(D)u(t-h)$$

for any given initial conditions for any initial conditions $y^{(i)}(0) = y_{i0}$; $i = 0,1,\ldots,n$. Note that the above theorem is useful to describe the presence of internal, i.e. in the state, time delays by observing that a sate vector taken with the output by incorporating its appropriate time-derivatives will have present the delay in the state components.

Theorem 3. let a transfer function in the simplest particular case be $G_{hh'}(s) = G'_{hh'}(s)/s$ with $h \neq h'$, where $G'_{hh'}(s) = \frac{N_0(s) + e^{-hs}N_d(s)}{D_0(s) + e^{-h's}D_d(s)}$ with $h' \neq h$. such that $d_{01} = \min_{\omega \in \boldsymbol{R}_{0+}} Re\left[-\boldsymbol{i}G_0(-\boldsymbol{i}\omega)\right]$, $N_0(s)/D_0(s)$ and $N_d(s)/D_d(s)$ are biproper, and

$$d_{01} > \sup_{\omega \in \boldsymbol{R}_{0+}} \left|\frac{N_d(\boldsymbol{i}\omega) - e^{-(h'-h)\boldsymbol{i}\omega}D_d(\boldsymbol{i}\omega)G_0(\boldsymbol{i}\omega)}{D_0(\boldsymbol{i}\omega) + e^{-\boldsymbol{i}h'\omega}D_d(\boldsymbol{i}\omega)}\right|.$$

The following properties hold:

(i) If $D_d(s)/D_0(s)$ is strictly bounded real (i.e. it is in $\boldsymbol{RH}_\infty$ with real coefficients and with H_∞ norm being strictly less than unity) then $p\left(s, e^{-h's}\right)$ is Hurwitz independent of h and irrespective of the size of h' and $G_{hh} \in \{SSPR\}$ for the given delays $h \geq 0$ and $h' \geq 0$.

(ii) If $D_0(s)/D_d(s)$ is strictly bounded real (i.e. it is in $\boldsymbol{RH}_\infty$ with real coefficients and with H_∞ norm being strictly less than unity) then $p\left(s, e^{-h's}\right)$ is Hurwitz independent of h and irrespective of the size of h' and $G_{hh'} \in \{SSPR\}$ for the given delays $h \geq 0$ and $h' \geq 0$.

(iii) $G_{hh'} \in \{SSPR\}$ is guaranteed in Properties [(i) -(ii)] irrespective of the delay size provided that

$$d_{01} > \sup_{\omega \in \boldsymbol{R}_{0+}} \frac{\left|N_d(\boldsymbol{i}\omega)\right| + \left|D_d(\boldsymbol{i}\omega)G_0(\boldsymbol{i}\omega)\right|}{\left|\left|D_0(\boldsymbol{i}\omega)\right| - \left|D_d(\boldsymbol{i}\omega)\right|\right|} \qquad \square$$

Proof: From Property 11, $G'_{hh'}(s) \in \{SPR_0\}$ if $G'_{hh'}(s)$ is stable and biproper with $Re\left(\boldsymbol{i}G'_{hh'}(\boldsymbol{i}\omega)\right) < 0$; $\forall \omega \in \boldsymbol{R}_{0+}$. The last condition can read equivalently

$$Re\left(\boldsymbol{i}G'^{*}_{hh'}(\boldsymbol{i}\omega)\right) = Im\ G'_{hh'}(\boldsymbol{i}\omega) = -Re\left(\boldsymbol{i}G'_{hh'}(\boldsymbol{i}\omega)\right) > 0$$

; $\forall \omega \in \boldsymbol{R}_{0+}$. Thus,

$$Re\left(\boldsymbol{i}G'^{*}_{hh'}(\boldsymbol{i}\omega)\right) = Re\left(\boldsymbol{i}\frac{N_0(\boldsymbol{i}\omega) + e^{-\boldsymbol{i}\omega h}N_d(\boldsymbol{i}\omega)}{D_0(\boldsymbol{i}\omega) + e^{-\boldsymbol{i}\omega h'}D_d(\boldsymbol{i}\omega)}\right)^{*}$$

$$= -Re\left(\boldsymbol{i}\frac{N_0(-\boldsymbol{i}\omega) + e^{\boldsymbol{i}\omega h}N_d(-\boldsymbol{i}\omega)}{D_0(-\boldsymbol{i}\omega) + e^{\boldsymbol{i}\omega h'}D_d(-\boldsymbol{i}\omega)}\right) > 0 \tag{18}$$

Define:

$$N_{01}(\boldsymbol{i}\omega) = -\boldsymbol{i}N_0(-\boldsymbol{i}\omega) = \left(\boldsymbol{i}N_0(\boldsymbol{i}\omega)\right)^{*} \quad D_{01}(\boldsymbol{i}\omega) = -\boldsymbol{i}D_0(-\boldsymbol{i}\omega) = \left(\boldsymbol{i}D_0(\boldsymbol{i}\omega)\right)^{*}$$

$$N_{d1}(\boldsymbol{i}\omega) = -\boldsymbol{i}N_d(-\boldsymbol{i}\omega) = \left(\boldsymbol{i}N_d(\boldsymbol{i}\omega)\right)^{*} \quad D_{d1}(\boldsymbol{i}\omega) = -\boldsymbol{i}D_d(-\boldsymbol{i}\omega) = \left(\boldsymbol{i}D_d(\boldsymbol{i}\omega)\right)^{*} \tag{19}$$

; $\forall \omega \in \boldsymbol{R}_{0+}$, so that $e^{-\boldsymbol{i}\omega h}N_{d1}(\boldsymbol{i}\omega) = \left(\boldsymbol{i}e^{\boldsymbol{i}\omega h}N_d(\boldsymbol{i}\omega)\right)^{*}$ and $e^{-\boldsymbol{i}\omega h}D_{d1}(\boldsymbol{i}\omega) = \left(\boldsymbol{i}e^{\boldsymbol{i}\omega h}D_d(\boldsymbol{i}\omega)\right)^{*}$; $\forall \omega \in \boldsymbol{R}_{0+}$

and then

$$Re\left(\boldsymbol{i}G'^{*}_{hh'}(\boldsymbol{i}\omega)\right) = Re\left(\frac{N_{01}(\boldsymbol{i}\omega) + e^{-\boldsymbol{i}\omega h}N_{d1}(\boldsymbol{i}\omega)}{D_{01}(\boldsymbol{i}\omega) + e^{-\boldsymbol{i}\omega h'}D_{d1}(\boldsymbol{i}\omega)}\right); \ \forall \omega \in \boldsymbol{R}_{0+}. \tag{20}$$

As a result, the proof of Theorem 3 follows as that of Theorem 2 under the replacements:

$$N_0(\boldsymbol{i}\omega) \to N_{01}(\boldsymbol{i}\omega) \ ; \ D_0(\boldsymbol{i}\omega) \to D_{01}(\boldsymbol{i}\omega)$$
$$N_d(\boldsymbol{i}\omega) \to N_{d1}(\boldsymbol{i}\omega) \ ; \ D_d(\boldsymbol{i}\omega) \to D_{d01}(\boldsymbol{i}\omega) \tag{21}$$

; $\forall \omega \in \boldsymbol{R}_{0+}$ by noting that

$$d_{01} > \sup_{\omega \in \boldsymbol{R}_{0+}} \left| \frac{N_d(\boldsymbol{i}\omega) - e^{-(h'-h)\boldsymbol{i}\omega}D_d(\boldsymbol{i}\omega)N_0(\boldsymbol{i}\omega)/D_0(\boldsymbol{i}\omega)}{D_0(\boldsymbol{i}\omega) + e^{-\boldsymbol{i}h'\omega}D_d(\boldsymbol{i}\omega)} \right|$$

is the same condition as

$$d_{01} > \sup_{\omega \in \boldsymbol{R}_{0+}} \left| \frac{N_{d1}(\boldsymbol{i}\omega) - e^{-(h'-h)\boldsymbol{i}\omega}D_{d1}(\boldsymbol{i}\omega)N_{01}(\boldsymbol{i}\omega)/D_{01}(\boldsymbol{i}\omega)}{D_{01}(\boldsymbol{i}\omega) + e^{-\boldsymbol{i}h'\omega}D_{d1}(\boldsymbol{i}\omega)} \right|$$

4. Numerical Simulations

This section contains some numerical examples illustrating the results introduced in the previous Section 3, in particular concerning Theorems 1 and 2. To this end, consider the transfer function (1) with the polynomials given by:

$$N_0(s) = 42(s+2), \ D_0(s) = s^2 + 13s + 42, N_d(s) = 3, \ D_d(s) = 2s + 1.$$

The time delay is given by $h = 2.5\,\text{s}$ while $d_0 = \min_{\omega \in R_{0+}} \operatorname{Re} G_0(i\omega) = 0.656$. Moreover, both $G_0(s)$ and $G_d(s)$ are proper, and

$$d_0 = 0.656 > 0.6031 = \sup_{\omega \in R_{0+}} |G_{ins}(i\omega)| = \sup_{\omega \in R_{0+}} \left| \frac{N_d(i\omega) - D_d(i\omega)G_0(i\omega)}{D_0(i\omega) + e^{-ih\omega} D_d(i\omega)} \right|.$$

Figure 1 shows the frequency response of the instrumental transfer function $G_{ins}(i\omega) = \frac{N_d(i\omega) - D_d(i\omega)G_0(i\omega)}{D_0(i\omega) + e^{-ih\omega} D_d(i\omega)}$ marking its peak value at 0.6031.

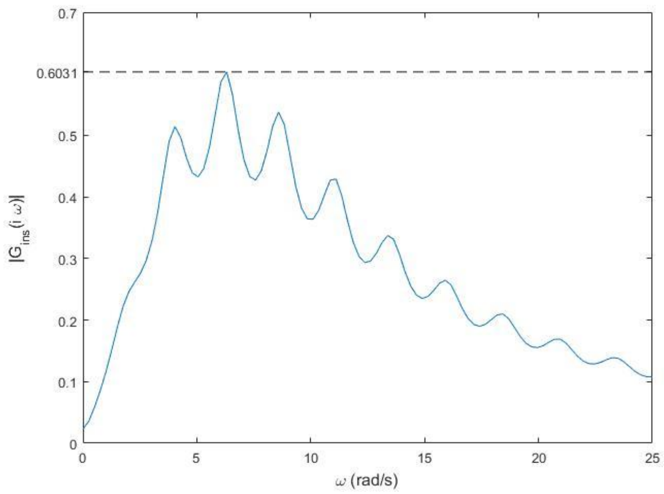

Fig. 1. Frequency response of $G_{ins}(i\omega)$ with its maximum value.

Now, we are in conditions of applying Theorem 1. In this way, the transfer function

$$\frac{D_d(s)}{D_0(s)} = \frac{2s+1}{s^2 + 13s + 42}$$

is strictly bounded real since it is stable with stable inverse while $\left\| \frac{D_d(s)}{D_0(s)} \right\|_\infty = 0.1539 < 1$. As a consequence, $p(s, e^{-hs}) = D_0(s) + e^{-hs} D_d(s)$ is Hurwitz regardless of the delay and $G_h \in \{SSPR\}$ for the delay $h = 2.5$ secs., according to Theorem 1*(i)*. Furthermore, we have that condition (3) holds since

$$d_0 = 0.656 > 0.6048 = \sup_{\omega \in R_{0+}} \frac{|N_d(i\omega) - D_d(i\omega)G_0(i\omega)|}{\big||D_0(i\omega)| - |D_d(i\omega)|\big|},$$

and we can conclude from Theorem 3*(iii)* that $G_h \in \{SSPR\}$ regardless of the delay and not only for the particular case of $h = 2.5$ secs. In order to check that $p(s, e^{-hs})$ is Hurwitz we may consider the equation $p(s, e^{-hs}) = D_0(s) + e^{-hs} D_d(s) = 0$, implying $1 + \frac{D_d(s)}{D_0(s)} e^{-hs} = 0$, and plot the Nyquist diagram of the transfer function $H(s) = \frac{D_d(s)}{D_0(s)} e^{-hs}$. Since $D_0(s)$ is Hurwitz, so will be $p(s, e^{-hs})$ if the Nyquist plot of $H(s)$ does not encircle the point (-1, 0). The following Figure 2 displays the Nyquist plot of $H(s)$ for two different values of the delay. It can be observed that the point (-1, 0) is not encircled any time (in fact, this point does not even appear in the figure since it would be displaced too much to the left) and $p(s, e^{-hs})$ is Hurwitz for the two delays considered as example, as Theorem 1 predicts.

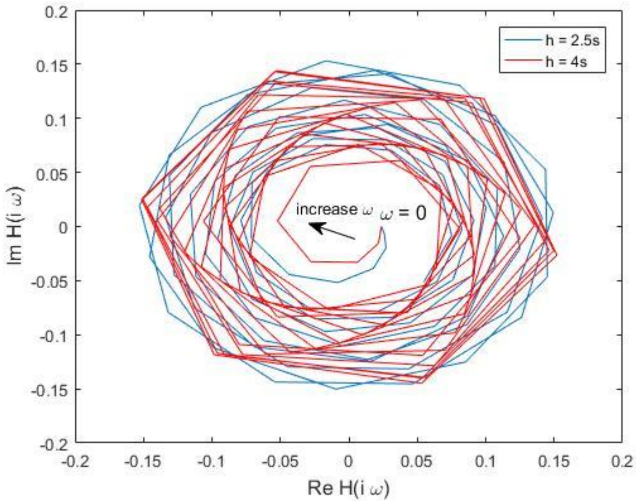

Fig. 2. Nyquist plot of $H(s)$ for two different values of the delay.

Now, consider the transfer function defined by Eq. (16) with the same polynomials as before and $h' = 4$ secs. . We again have $d_0 = \min_{\omega \in R_{0+}} Re\, G_0(\boldsymbol{i}\omega) = 0.656$, both $G_0(s)$ and $G_d(s)$ are proper, and

$$d_0 = 0.656 > 0.6437 = \sup_{\omega \in R_{0+}} |G_{ins2}(\boldsymbol{i}\omega)| = \sup_{\omega \in R_{0+}} \left| \frac{N_d(\boldsymbol{i}\omega) - e^{-(h'-h)\boldsymbol{i}\omega} D_d(i\omega) G_0(\boldsymbol{i}\omega)}{D_0(\boldsymbol{i}\omega) + e^{-ih'\omega} D_d(\boldsymbol{i}\omega)} \right|.$$

Figure 3 depicts the frequency response of the instrumental transfer function $G_{ins2}(\boldsymbol{i}\omega) = \frac{N_d(\boldsymbol{i}\omega) - e^{-(h'-h)i\omega} D_d(\boldsymbol{i}\omega) G_0(\boldsymbol{i}\omega)}{D_0(\boldsymbol{i}\omega) + e^{-ih'\omega} D_d(\boldsymbol{i}\omega)}$ marking its peak value at 0.6437.

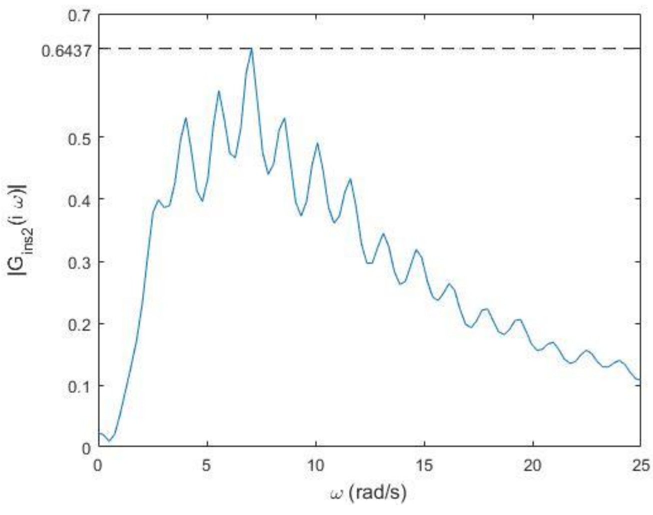

Fig. 3. Frequency response of $G_{ins2}(\boldsymbol{i}\omega)$ with its maximum value.

Thus, $D_d(s)/D_0(s)$ is strictly bounded real and from Theorem 2*(i)* we can conclude that $p(s, e^{-h's})$ is Hurwitz regardless the values of the delays *h* and *h'*. This fact is corroborated by the Nyquist plot of

$H'(s) = \dfrac{D_d(s)}{D_0(s)} e^{-h's}$ in the same way as before. The following Fig. 4 displays the Nyquist plot of $H'(s)$ for $h' = 4$ secs. and $h' = 6$ secs., as example. It can be deduced from Fig. 4 that the point (-1,0) is not encircled any time resulting in a stable polynomial for any value of the delay, as Theorem 2 claims. In addition, we may test the stability of $G_{hh'}(s)$ by applying a unity step input to it. Thus, Fig. 5 displays the step response of $G_{hh'}(s)$ for different values for the delays. The figure contains the cases when $h = h' = 0$, $h = 2.5$ secs. and $h' = 4$ s, and $h = 9$ secs. and $h' = 5$ secs.. It can be observed from Fig. 5 that the output is bounded and converges to a constant value in all cases, implying the stability of $G_{hh'}(s)$, as concluded from Theorem 2.

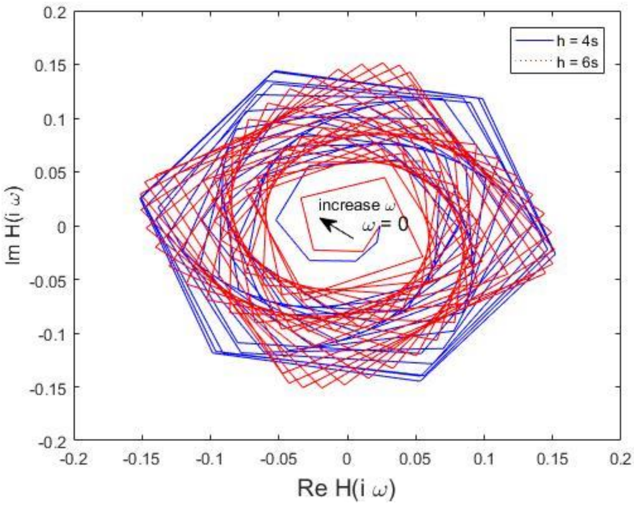

Fig. 4. Nyquist plot of $H'(s)$ for two different values of the internal delay.

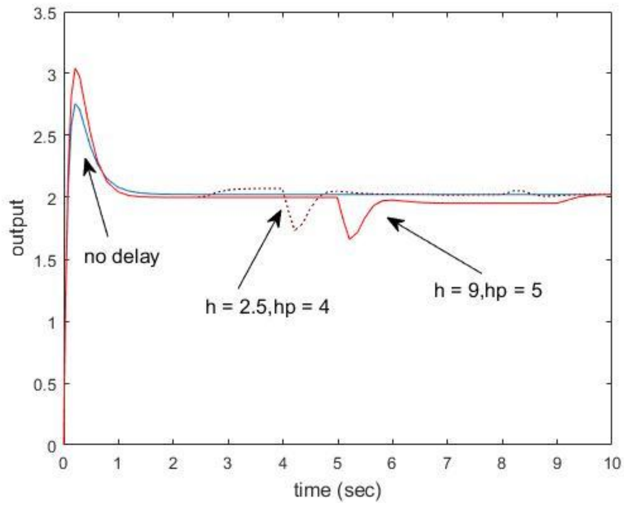

Fig. 5. Step response for $G_{hh'}(s)$ and various internal and external delays h and $h' = hp$.

Furthermore, from Theorem 2*(i),* we can also deduce that $G_{hh'} \in \{SSPR\}$ for the particular iternal and external delays $h = 2.5$ secs. and $h' = 4$ secs.. However, in this case the condition *(iii)* of Theorem 2 is not satisfied since

$$d_0 = 0.656 < 0.6629 = \sup_{\omega \in R_{0+}} \frac{|N_d(i\omega)| + |D_d(i\omega)G_0(i\omega)|}{\left\| D_0(i\omega)| - |D_d(i\omega) \right\|}$$

Consequently, $G_{hh'} \notin \{SSPR\}$ independently of the delays and Theorem 2 inform us that the $\{SSPR\}$ condition must be checked every time for every pair of delay values. Finally, Fig. 6 depicts the polar plot of the frequency response of $G_{hh'}(i\omega)$ for the delay-free and the case of $h = 2.5$ secs. and $h' = 4$ secs. . It can be observed that the presence of delays visibly modifies the frequency response of the system, fact that points out the complexity of dealing with time-delay systems. In this way, the presented methods offer a valuable tool to study the stability and sensitivity with respect to delay of such systems.

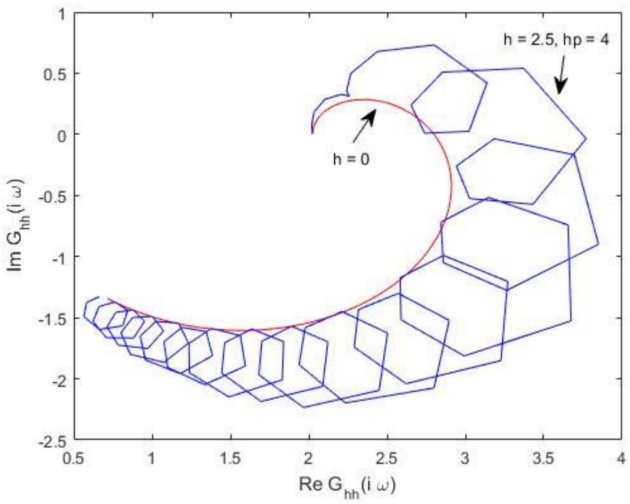

Fig. 6. Frequency responses of $G_{hh'}(i\omega)$ without delay and for $h = 2.5$ secs. and $h' = hp = 4$ secs.

5. Some Applicability Discussion

We pay now attention to two control configurations displayed in block diagrams in Figs. 7 and 8 below which involve the presence of point delays and which adjust to the theoretical formalism of the above sections. We also present later on a physical example concerning positive realness which adjusts to the numerical examples discussed in the above section.

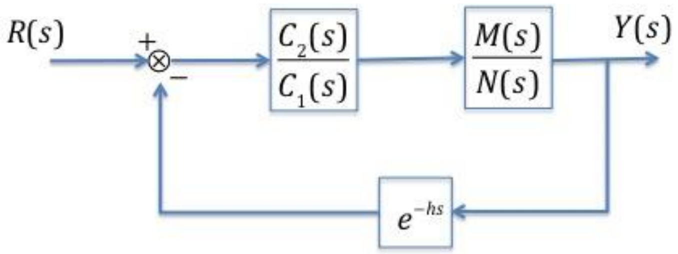

Fig. 7. Closed-loop control system with delayed feedback.

The proposed methods can used to analyse the stability of feedback control loops involving time delays in a variety of situations. For instance, Fig. 7 displays a closed-loop control system with a delayed feedback. These systems arise, for example, when a sensor delay appears in the output measurement system. The closed-loop transfer function is:

$$\frac{Y(s)}{R(s)} = \frac{M(s)C_2(s)}{N(s)C_1(s) + M(s)C_2(s)e^{-hs}}$$

which is a particular case of (1). In this way, Theorems 1-3 can be used to analyse the stability of the closed-loop system in the presence of the sensor delay along the potential dependency of stability with delay (sensitivity analysis).

Thus, the above closed-loop configuration could play the role of a feed-forward block which has a transfer functions which adjusts to the theoretical framework of the above sections and contain delays. If a minimum input-output interconnection gain to achieve the requirements on strict positive realness is needed then such a gain could be incorporated from the input r to the output y. If the resulting transfer function, subject eventually to delays, satisfy the requirements for strong strict positive realness of the given theoretical framework then any controller (being, perhaps, nonlinear and time-varying) belonging to any hyperstable class of controllers being defined as the class which satisfies a Popov's type inequality of the form below:

$$\int_0^t r(\tau)(-y(\tau))d\tau \geq -\gamma \text{ for all } t > 0$$

and any given finite positive real constant γ. Any controller within such a class is valid to generate the controlled plant feedback input $r(t)$ to the feed-forward controlled plant . The whole configuration of controlled plant (feed-forward loop) and any controller of the given hyperstable class gives as a result that $0 < E(t) \leq \gamma$, where $E(t) = \int_0^t r(\tau)y(\tau)d\tau$ is the input-output energy for any $t > 0$. See [1,4, 7, 16]. The inequality that the energy is positive comes from the positivity of the feed-forward loop which is the central framework in this paper. The boundedness of the energy comes from Popov's inequality being satisfied by any controller in such a class. The final result is that the closed-loop system is globally asymptotically hyperstable, that is, globally asymptotically stable for a feed-forward plant, with eventual point delays, which is strongly strictly positive real and any controller which satisfies Popov's inequality.

Other alternative control schemes can also be represented in closed-loop form by equations that are particular cases of (1). For example, Figure 8 displays other example when the plat itself possesses a delay.

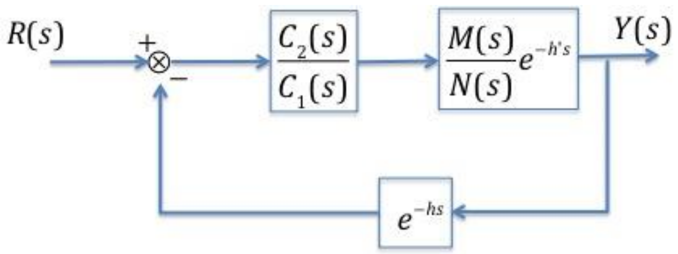

Fig. 8. Closed-loop control of a delayed system with delayed feedback.

In this case, the transfer function is:

$$\frac{Y(s)}{R(s)} = \frac{M(s)C_2(s)}{N(s)C_1(s) + M(s)C_2(s)e^{-hs}} e^{-h's}$$

since the term $e^{-h's}$ appears in a multiplicative way, since it is an external delay (i.e. either in the input or the output), it does not threaten the stability of the closed-loop and can be omitted in the subsequent stability and delay sensitivity analysis. Consequently, the methods introduced in this paper could be applied to the transfer function, which is also a particular case of (1). In this way, the proposed approach is revealed as a method with important applications to Engineering Systems.

We now discuss the positive realness issues of a simple practical example with engineering insight which is worked from an analytic point of view. The involved transfer function is subject to feedback and includes a delay. This example is a practical case study which adjusts to the discussions of Section 4 and which can be interpreted as several alternative possible particular cases of the mathematical results of Section 3.

Consider a series RLC electric circuit which has a tandem of a voltage source and a current source described by the equation:

$$v(t)+Ki(t)=L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int_0^t i(\tau)d\tau+v_c(0)$$

In particular, the current-dependent voltage and the resistor R are equivalent to a whole resistor of value $(R-K)$ which is positive if $R>K$, null if both are identical and negative if $K>R$. This delay-free circuit is discussed in [23]. See also, for instance, [24-26] for other implementations involving positivity or passivity issues. Assume that a transformer of gain d is coupled in series with a current-dependent voltage source of gain λ, the current being that generated from the above part of the circuit subject to a constant delay h giving the total tandem voltage:

$$v_r(t)=dv(t)+\lambda i(t-h)$$

The above circuitry is displayed in Fig. 9 below:

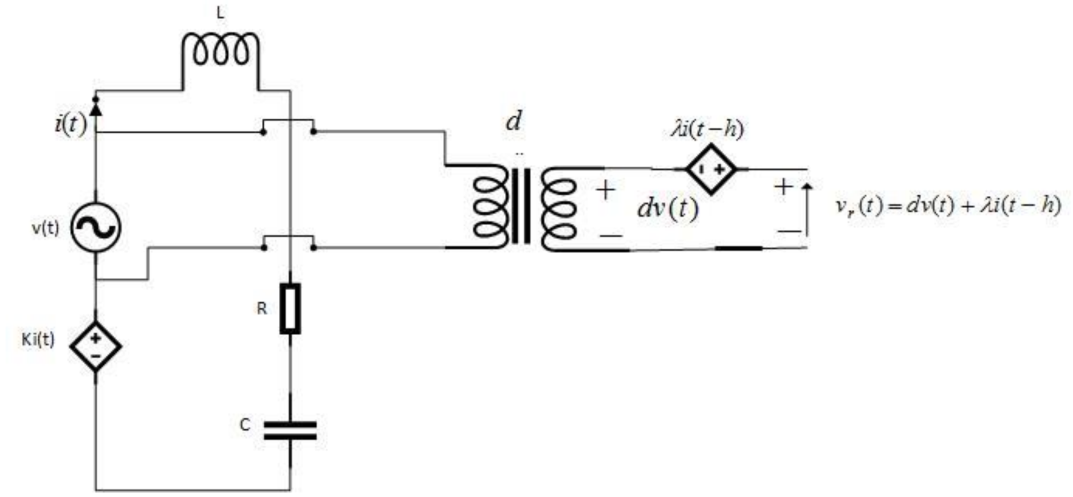

Fig. 9. Electric circuit with delay.

It is assumed that λ and d are independent of the frequency in a wide range of frequencies of interest. Now, take Laplace transforms in the first equation with zero initial conditions and replace $V(s)$, the Laplace transform of $v(t)$, in the second one to yield:

$$G(s)=\frac{V_r(s)}{V(s)}=\frac{\lambda Cse^{-hs}}{LCs^2+(R-K)Cs+1}+d \tag{22}$$

Note that the delay h is an external delay in the above characterization. Define auxiliary parameters $k=\lambda/L$, $a=(R-K)/L$, $b=1/LC$. Two cases are discussed for zero and non-zero delays:

a) *Zero delay*: If $R>K$ (then $a>0$), $d=0$ (i.e. null input-output direct interconnection gain) and $h=0$ (i.e. delay-free case). Note that the transfer function $G(s)$ has relative degree one, it is stable and $Re\ G(\boldsymbol{i}\omega)=\frac{ka\omega^2}{(b-\omega^2)^2+a^2\omega^2}\geq 0$; $\forall\omega\in\boldsymbol{R}_{0+}$ for any $\lambda>0$, then $a>0$ and $b>0$. Then, $G\in\{PR\}$. Also, $G\notin\{SPR\}$ since the hodograph tends to zero as the frequency tends to $\pm$ infinity and it is also zero for zero frequency. The polar and Nyquist frequency plots are tangent with the zero point of the complex plane at infinity frequencies. Since, at finite frequencies the polar plot lies fully in the fourth quadrant of the complex plane, one concludes positive realness. Since the poles are both strictly stable, $G\in\{PR\}$. Note that any positive input-output interconnection gain makes $G\in\{SSPR\}$ since

$$Re\, G(\boldsymbol{i}\omega)=\frac{ka\,\omega^2}{\left(b-\omega^2\right)^2+a^2\omega^2}+d>0\,;\ \forall\omega\in\boldsymbol{R}_{0+}\,.$$

b) *Non-zero delay*: Now,

$$G(\boldsymbol{i}\omega)=\frac{\boldsymbol{i}k\omega(cos(h\omega)-\boldsymbol{i}(sinh\omega))}{\left(b-\omega^2\right)+\boldsymbol{i}a\omega}+d=\frac{k\omega\, sin\,(h\omega)+\boldsymbol{i}k\omega cos(h\omega)}{\left(b-\omega^2\right)+\boldsymbol{i}a\omega}+d$$

$$=\frac{\left[\left(b-\omega^2\right)-\boldsymbol{i}a\omega\right]\,\left[k\omega\, sin\,(h\omega)+\boldsymbol{i}k\omega cos(h\omega)\right]}{\left(b-\omega^2\right)^2+a^2\omega^2}+d$$

Then,

$$Re\,G(\boldsymbol{i}\omega)=k\omega\frac{\left(b-\omega^2\right)sin\,(h\omega)+a\omega cos\,(h\omega)}{\left(b-\omega^2\right)^2\ +a^2\omega^2}+d$$

Some lower-bounds of $\left|Re\,G(\boldsymbol{i}\omega)\right|$ for all $\omega\in cl\boldsymbol{R}\left(=\boldsymbol{R}\cup\{\pm\infty\}\right)$ are:

$$Re\,G(\boldsymbol{i}\omega)\geq \inf_{\omega\in cl\boldsymbol{R}_{0+}} Re\,G(\boldsymbol{i}\omega)\geq d-d_1\,;\ d_1=\sqrt{2}k\max_{\omega\in cl\boldsymbol{R}_{0+}} d_{1a}(\omega)$$
$$Re\,G(\boldsymbol{i}\omega)\geq \inf_{\omega\in cl\boldsymbol{R}_{0+}} Re\,G(\boldsymbol{i}\omega)\geq d-d_2\,;\ d_2=k\max_{\omega\in cl\boldsymbol{R}_{0+}} d_{2a}(\omega)$$
$$Re\,G(\boldsymbol{i}\omega)\geq \inf_{\omega\in cl\boldsymbol{R}_{0+}} Re\,G(\boldsymbol{i}\omega)\geq d-d_3\,;\ d_3=k\max_{\omega\in cl\boldsymbol{R}_{0+}} d_{3a}(\omega)$$

where $cl\,\boldsymbol{R}_{0+}=\boldsymbol{R}_{0+}\cup\{+\infty\}$, and

$$d_{1a}(\omega)=\left(\frac{\omega\left|b-\omega^2\right|}{\left(b-\omega^2\right)^2\ +a^2\omega^2},\frac{a\omega^2}{\left(b-\omega^2\right)^2\ +a^2\omega^2}\right),\ d_{2a}(\omega)=\frac{\omega\left|b-\omega^2\right|\left|sin\,(h\omega)\right|+a\omega^2\sqrt{1-sin^2(h\omega)}}{\left(b-\omega^2\right)^2\ +a^2\omega^2},$$

$$d_{3a}(\omega)=\frac{\omega\left|b-\omega^2\right|\sqrt{1-cos^2(h\omega)}+a\omega^2\left|cos(h\omega)\right|}{\left(b-\omega^2\right)^2\ +a^2\omega^2}.$$

Note that the symmetry of the hodograph allows to make feasible checking the inequalities just for positive frequencies including $\{+\infty\}$ and $\omega=0$. Note also that d_1, d_2 and d_3 are finite. So, $G(s)$ is strongly positive real if the interconnection input-output gain d is large enough to exceed strictly any of the values d_1, d_2 or d_3. In particular, the condition $d>d_1$ guarantees that $G\in\{SSPR\}$ independent of the size of the delay h. The conditions of minimum input-output interconnection gain $d>d_2$ or $d>d_3$ can be tested for strong positive realness being dependent on the delay size h. The calculations of d_i for $i=1,2,3$ can be performed;

a) from the plots of the functions versus frequency $d_{ia}(\omega)$ for $i=1,2,3$;

b) by calculating analytically the maxima of $d_{ia}(\omega)$ for $i=1,2,3$;

c) by implementing the following steps: c1) displaying the polar frequency plot of the delay-free transfer function, c2) correcting a set of sufficiently tight set of points of such a frequency plot with a circumference or radius one centred at each of such points which takes account for the delay modification of the polar plot, c3) establish and estimation of the minimum value of the real part of the hodograph $d_{min}\leq \inf_{\omega\in cl\boldsymbol{R}_{0+}} Re\,G(\boldsymbol{i}\omega)$ (this value could be negative), c4) the input-output interconnection gain satisfying the constraint $d>\left|d_{min}\right|$ guarantees that $G\in\{SSPR\}$.

The above design of the input-output interconnection gain guarantees that the hodograph associated with the transfer function lies in the first and fourth quadrants of the complex plane and they should be applied only if the transfer function is strictly stable which is a necessary condition for its strong strict positive realness. Otherwise, the transfer function should be first stabilized via some stabilizing controller and then to ensure its strong positive realness by the incorporation of a sufficiently large positive input-output interconnection gain. Note that the input $(v(t))$-output $(v_r(t))$ energy is positive for all $t>0$ as a direct consequence of the strong strict positive realness of the transfer function. Finally, note also that the properties of positive realness of the transfer function (22) are covered in a formal setting by any of the following particular cases:

a) Theorem 1 with $N_0 = D_d = 0$;

b) Theorems 2 and 3 with $N_0 = 0$ and $h' = 0$ while D_0 and D_d are non-uniquely selected so that its sum equalizes the denominator polynomial of the current transfer function;

c) Theorems 2 and 3 with $N_0 = D_d = 0$ while D_0 equalizes the denominator polynomial.

6. Conclusions

The paper has discussed the fact that if the real positivity of a transfer function in the delay-free case is not strict then the property is usually lost in the presence of delays. Also, it has been studied the case when the transfer function is composed of a parallel time-delay free one with one being a subject to an external delay which is physically present in either the input or the output of the dynamic system described by the second part of a such a parallel disposal. This section has also included other various main results for the case when the transfer function is described by a quotient of quasi-polynomials composed of additive delay-free and delayed terms. Such a description is, in particular, useful to describe the case of internal (i.e. in the state) delays. The study has also included a further result which considers the combined presence of internal and external delays. Some applicability discussion and numerical results have also been given, with illustrative frequency responses, Nyquist plots and time responses and a worked physical example involving circuitry, which are concerned with the studied properties of positive realness transfer functions subject to point delays.

Acknowledgements

The authors are very grateful to the Spanish Government and to the European Fund of Regional Development FEDER for Grants DPI2015-64766-R and DPI2016-77271-R and to UPV/EHU for Grant PGC 17/33. They are also grateful to the reviewers for their suggestions to improve the two former versions of the manuscript.

References

[1] M. De la Sen, "Stability of composite systems with an asymptotically hyperstable system," International Journal of Control, vol. 44, no. 6, pp. 1769-1775, 1986.

[2] A. Megretski, "Lyapunov Functions and Storage Functions," Massachusetts Institute of Technology, Lecture 5, 6.243j, Fall 2003, 2003.

[3] A. Fradkov, "Passification of non-square linear systems and feedback Yakubovich-Kalman-Popov Lemma," *European Journal of Control*, vol. 6, pp. 577-586, 2003.

[4] V. M. Popov, *Hyperstability of Automatic Control Systems*. New York: Springer, 1973.

[5] W. P. Heath, J. Carrasco, and M. De la Sen, "Second-order counterexample to the discrete-time Kalman conjecture," *Automatica*, vol. 60, pp. 140-144, 2015.

[6] M. De la Sen, "Preserving positive realness through discretization," *Positivity*, vol. 6, no. 1, pp. 31-45, 2002.

[7] C. Batlle, "Passive control theory. Parts I-II," presented at *II Eurocon/Geoplex Summer School on Modelling and Control of Complex Dynamic Systems*, Bernitoro, Italy, July 18-22, 2005.

[8] Y. D. Landau, *Adaptive Control: The Model Reference Approach*, J. M. Mendel Ed. New York: Marcel Dekker, 1979.

[9] M. De la Sen, "Sufficiency-type stability and stabilization criteria for linear time-invariant systems with constant point delays," *Acta Applicandae Mathematicae*, vol. 83, no.3, pp. 235-256, 2004.

[10] M. De la Sen, "The generalized Beverton-Holt equation and the control of populations," Applied Mathematical Modelling, 2008, vol. 32, no 11, p. 2312-2328.

[11] M. De la Sen and A. Ibeas, "Stability results for switched linear systems with constant discrete delays," *Mathematical Problems in Engineering*, Article ID 543145, vol. 2008, 2008.

[12] M. De la Sen and S. Alonso-Quesada, "Model-matching-based control of the Beverton-Holt equation in ecology," *Discrete Dynamics in Nature and Society*, vol. 2008, Article ID 793512, 2008.

[13] T. Nampradit and D. Banjerdpongchai, "On computing the worst-case H-infinity performance of Luré systems with uncertain time delays," *Engineering Journal-Thailand*, vol. 19, no. 5, pp. 101-120, 2015.

[14] N. Raksasri and J. Songsiri, "Guaranteed stability of autoregressive models with Granger causality learned from Wald tests," *Engineering Journal-Thailand*, vol. 21, no. 6, pp. 23-36, 2017.

[15] T. Tanaka and C. Langbort, "The bounded real lemma for internally positive systems and H-infinity structured static state feedback," *IEEE Transactions on Automatic Control*, vol. 56, no. 9, pp. 2218-2223, 2011.

[16] J. H. Taylor, "Strictly positive-real functions and the Lefschetz-Kalman-Yakubovich (LKY) Lemma," *IEEE Transactions on Circuits and Systems*, vol. 21, no. 2, pp. 310- 311, 1974.

[17] M. Alberdi, M. Amundarain, A. J. Garrido, I. Garrido, O. Casquero, and M. De la Sen, "Complementary control of oscillating water column-based wave energy conversion plants to improve the instantaneous power output," *IEEE Transactions on Energy Conversion*, vol. 26, no. 4, pp. 1033- 1040, 2011.

[18] V. M. Marchenko, "Hybrid discrete-continuous systems: I. stability and stabilizability," *Differential Equations*, vol. 48, no. 12, pp. 1623-1638, 2012.

[19] G. Rajchakit, T. Rojsiraphisal, and M. Rajchakit, "Robust stability and stabilization of uncertain switched discrete- time systems," *Advances in Difference Equations*, vol. 2012, no. 134, 2012.

[20] X. Z. Liu and P. Stechlinski, "Hybrid stabilization and synchronization of nonlinear systems with unbounded delays," *Applied Mathematics and Computation*, vol. 280, pp. 140-161, 2016.

[21] G. Lu, L. Yeung, D. Ho, and Y. Zheng, "Strict positive realness for linear time-invariant systems with time-delays," in *39th Proc. Conf. Decision and Control*, 2000, vol. 5, pp. 5004-5009.

[22] E. Fridman and U. Shaked, "On delay-dependent passivity," *IEEE Transactions on Automatic Control*, vol. 47, no. 4, pp. 664-669, 2002.

[23] P. H. Lewis and C. Yang, *Basic Control Systems Engineering*. Englewood Cliffs: Prentice Hall, 1997.

[24] P. Suriyavejwongs, E. Leelarasmee, and W. Pora, "A zero bias pixel sensor and its zero-bias column buffer-direct-injection circuit," *Engineering Journal-Thailand*, vol. 21, no. 5, pp. 178-191, 2017.

[25] X. N. Song, S. Song, and I.T. Balsera, "Mixed H-infinity/passive projective synchronization for fractional-order neural networks with uncertain parameters and delays," *International Journal of Innovative Computing Information and Control*, vol. 13, no. 4, pp. 1273-1288, 2017.

[26] C. Jettanasen, "Modeling and optimizing a passive EMI filter to attenuate a common-mode current at high frequency in a three-phase adjustable-speed AC motor drive," *International Journal of Innovative Computing Information and Control*, vol. 8, no. 10A, pp. 6629-6642, 2012.